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MESSENGER OF MATHEMATICS.

THE CIRCLES THAT TOUCH ANY THREE OF THE FOUR CIRCLES TOUCHING THE SIDES OF A TRIANGLE.*

[With notes on Tucker Circles.]

By *T. C. Lewis, M.A.*

1. Let ABC be any triangle, and let a line parallel to BC meet AB in B' and AC in C'; and let $AB' = nB'B$ (*vide* fig. 1).

Using the quadri-circular coordinates of Gaston Darboux, if ABC be the triangle of reference, the coordinates of any point Q are x_1, x_2, x_3, x_4 and the radii of reference are $\rho_1, \rho_2, \rho_3, \rho_4$. With $AB'C'$ as the triangle of reference, let the coordinates of Q be x_1', x_2', x_3', x_4' and the radii of reference $\rho_1', \rho_2', \rho_3', \rho_4'$.

Then $$\rho_1' x_1' + \rho_1'^2 = \rho_1 x_1 + \rho_1^2,$$

therefore $$\rho_1' x_1' = \rho_1 x_1 + \frac{2n+1}{(n+1)^2}\rho_1^2.$$

And $$\rho_2' x_2' = QB'^2 - \frac{n^2}{(n+1)^2}\rho_2^2$$

$$= \frac{1}{n+1}\left(\rho_1 x_1 + n\rho_2 x_2 + \frac{\rho_1^2}{n+1}\right).$$

Similarly $$\rho_3' x_3' = \frac{1}{n+1}\left(\rho_1 x_1 + n\rho_3 x_3 + \frac{\rho_1^2}{n+1}\right),$$

and $$\rho_4' x_4' = \frac{1}{n+1}\left(\rho_1 x_1 + n\rho_4 x_4 + \frac{\rho_1^2}{n+1}\right).$$

* The problem here considered was dealt with by Mr. Henry M. Jeffery in the *Quarterly Journal*, vol. xxiii, pp. 180–197. I was not acquainted with his paper when I undertook the present investigation after reading a paper by Dr. John Dougall, to which I refer. The method I have adopted is different from that employed by Mr. Jeffery, and appears to throw further light on the geometry of the problem.

The equation to any circle through B', C' may be written

$$k(\rho_1'x_1' - \rho_4'x_4') + n(\rho_2'x_2' + \rho_3'x_3') = 0,$$

which may be transformed to coordinates referred to the triangle ABC by substituting for $\rho_1'x_1'$, $\rho_2'x_2'$, &c., the values already obtained. The equation thus becomes

$$k\{(n+1)(\rho_1x_1 - \rho_4x_4) + 2\rho_1^2\} + 2(n+1)\rho_1x_1 + n(n+1)(\rho_2x_2 + \rho_3x_3) + 2\rho_1^2 = 0,$$

which reduces to

$$\{n(2+k)+1\}\rho_1x_1 + \left\{n(n+1) - (1+k)\frac{\rho_1^2}{\rho_2^2}\right\}\rho_2x_2 + \left\{n(n+1) - (1+k)\frac{\rho_1^2}{\rho_3^2}\right\}\rho_3x_3 - \left\{(n+1)k + (1+k)\frac{\rho_1^2}{\rho_4^2}\right\}\rho_4x_4 = 0 \ldots\ldots (1).$$

Here the sum of the squares of the coefficients of x_1, x_2, x_3, x_4 is found to be

$$(n+1)^2(k^2a_1^2 + n^2a^2),$$

where a_1 is the distance of A from the orthocentre of the triangle. Thus if ρ is the radius of the circle under consideration

$$2(n+1)\rho = \sqrt{(k^2a_1^2 + n^2a^2)} \ldots\ldots (2).$$

2. If the circle (1), passing through the points B', C', touches the circle escribed on the side b of the triangle, the equation of which is

$$(a_1 - \tfrac{1}{2}r_2)\frac{x_1}{\rho_1} - (a_2 + \tfrac{1}{2}r_2)\frac{x_2}{\rho_2} + (a_3 - \tfrac{1}{2}r_2)\frac{x_3}{\rho_3} - \tfrac{1}{2}r_2\frac{x_4}{\rho_4} = 0,$$

then the following relation must be satisfied, viz.

$$\{n(1+k)+n+1\}(a_1 - \tfrac{1}{2}r_2) - \left\{n(n+1) - (1+k)\frac{\rho_1^2}{\rho_2^2}\right\}(a_2 + \tfrac{1}{2}r_2) + \left\{n(n+1) - (1+k)\frac{\rho_1^2}{\rho_3^2}\right\}(a_3 - \tfrac{1}{2}r_2) + \left\{(n+1)(1+k) - (n+1) + (1+k)\frac{\rho_1^2}{\rho_4^2}\right\}\tfrac{1}{2}r_2 = (n+1)\sqrt{(k^2a_1^2 + n^2a^2)},$$

i.e.

$$a_1 - n(a_2 - a_3) - (n+1)r_1 + \frac{1+k}{n+1}\left\{(n+1)a_1 - \frac{\rho_1^2}{r_1}\right\} = \sqrt{(k^2a_1^2 + n^2a^2)} \quad \ldots\ldots(3).$$

If at the same time the circle (1) touches the circle escribed on the side c, we should also have

$$a_1 + n(a_2 - a_3) - (n+1)r_1 + \frac{1+k}{n+1}\left\{(n+1)a_1 - \frac{\rho_1^2}{r_1}\right\} = \sqrt{(k^2a_1^2 + n^2a^2)} \quad \ldots\ldots(4),$$

the contact being of the same character (external or internal) in the two cases, *i.e.* the sign of the radical being the same.

If a circle (1) can touch the two escribed circles in the manner indicated, then for some value of k the relations (3) and (4) will be simultaneously satisfied. Each of them yields two values of k, one of which would be common to both.

By subtraction

$$1 + k = -(n^2 - 1)\frac{r_2 r_3}{\rho_1^2} \quad \ldots\ldots(5),$$

therefore $$2ka_1 = r_1 - r - n^2(r_2 + r_3),$$

and by addition

$$2a_1 - (n+1)(r_2 + r_3) + 2(1+k)a_1 - \frac{1+k}{n+1}\cdot\frac{r_2 + r_3}{r_2 r_3}\rho_1^2 = 2\sqrt{(k^2a_1^2 + n^2a^2)} \quad \ldots\ldots(6),$$

i.e.

$$r_2 + r_3 + r - r_1 - (n+1)(r_2 + r_3) + 2a_1\frac{n}{n+1}(1+k) = 2\sqrt{(k^2a_1^2 + n^2a^2)}.$$

Substitute the value of $(1+k)$ on the left side, therefore

$$-(r_1 - r) - n(r_2 + r_3) - n(n-1)(r_2 + r_3) = 2\sqrt{(k^2a_1^2 + n^2a^2)},$$

therefore $$-(r_1 - r) - n^2(r_2 + r_3) = 2\sqrt{(k^2a_1^2 + n^2a^2)},$$

therefore $$\{r_1 - r + n^2(r_2 + r_3)\}^2 = 4(k^2a_1^2 + n^2a^2),$$

and this is true for the same value of k, for

$$4k^2a_1^2 + 4n^2a^2$$

$$= (r_1 - r)^2 - 2n^2(r_2 + r_3)(r_1 - r) + n^4(r_2 + r_3)^2 + 4n^2(r_2 + r_3)(r_1 - r)$$

$$= \{r_1 - r + n^2(r_2 + r_3)\}^2.$$

Thus the value of k in (5) satisfies (6), *i.e.* it is common to (3) and (4). Therefore for any line $B'C'$ parallel to BC a circle can be drawn which passes through B', C' and touches the circles escribed on the sides b and c.

The equation to this circle is

$$\left\{1 - n(n-1)\frac{r_2 r_3}{\rho_1^2}\right\}\rho_1 x_1 + \left\{n + (n-1)\frac{r_2 r_3}{\rho_2^2}\right\}\rho_2 x_2$$
$$+ \left\{n + (n-1)\frac{r_2 r_3}{\rho_3^2}\right\}\rho_3 x_3$$
$$+ \left\{1 + (n^2-1)\frac{r_2 r_3}{\rho_1^2} + (n-1)\frac{r_2 r_3}{\rho_4^2}\right\}\rho_4 x_4 = 0 \ldots\ldots (7).$$

From (2) it follows that the radius is given by

$$4(n+1)\rho = r_1 - r + n^2(r_2 + r_3) \ldots\ldots\ldots\ldots (8),$$

the right-hand side to be taken negatively if $n+1$ is negative, to give a positive value for ρ.

3. The theorem is equally true if $B'C'$ be anti-parallel to BC with respect to the sides AB, AC. For if B_1, C_1 (fig. 1)

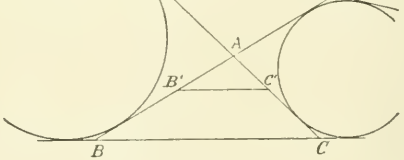

Fig. 1.

be the points in which BA and CA meet the fourth tangent to the two circles escribed on those sides, those circles are also escribed to the triangle AB_1C_1. Therefore, by the preceding proof, if $B'C'$ is parallel to B_1C_1, *i.e.* anti-parallel to BC with respect to AB, AC, a circle can be drawn through B', C' to touch those two escribed circles.

Dr. John Dougall* has already shown that Casey's condition that four circles should be tangible by a circle is fulfilled for the two escribed circles and the point circles at B' and C'. That method does not, however, determine the tangent circle. The present investigation was undertaken after reading Dr. Dougall's paper.

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* *Proceedings, Edin. Math. Soc*, March 1915.

4. We may now find the condition that the circle represented by (7) may also touch the circle escribed on the side a with contact of the same character as that which it has with the other two escribed circles.

The equation to the circle escribed on the side a is

$$-(a_1+\tfrac{1}{2}r_1)\frac{x_1}{\rho_1}+(a_2-\tfrac{1}{2}r_1)\frac{x_2}{\rho_2}+(a_3-\tfrac{1}{2}r_1)\frac{x_3}{\rho_3}-\tfrac{1}{2}r_1\frac{x_4}{\rho_4}=0,$$

and the required condition is

$$-\left\{1-n(n-1)\frac{r_2r_3}{\rho_1^2}\right\}(a_1+\tfrac{1}{2}r_1)+\left\{n+(n-1)\frac{r_2r_3}{\rho_2^2}\right\}(a_2-\tfrac{1}{2}r_1)$$
$$+\left\{n+(n-1)\frac{r_2r_3}{\rho_3^2}\right\}(a_3-\tfrac{1}{2}r_1)$$
$$-\left\{1+(n^2-1)\frac{r_2r_3}{\rho_1^2}+(n-1)\frac{r_2r_3}{\rho_4^2}\right\}\cdot\tfrac{1}{2}r_1=\sqrt{(k^2a_1^2+n^2a^2)}$$
$$=-\tfrac{1}{2}(r_1-r)-\tfrac{1}{2}n^2(r_2+r_3)\quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

this is divisible throughout by $n-1$, and we have

$$r+\tfrac{1}{2}(n+1)(r_2+r_3)+n\frac{r_2r_3}{\rho_1^2}(a_1+\tfrac{1}{2}r_1)+\frac{r_2r_3}{\rho_2^2}(a_2-\tfrac{1}{2}r_1)$$
$$+\frac{r_2r_3}{\rho_3^2}(a_3-\tfrac{1}{2}r_1)-\tfrac{1}{2}r_1r_2r_3\left\{(n+1)\frac{1}{\rho_1^2}+\frac{1}{\rho_4^2}\right\}=0,$$

therefore
$$n(r_2+r_3)=-r-\frac{r_2r_3}{r},$$

therefore
$$n=-\frac{as+bc}{as},$$

therefore
$$BB'=\frac{c}{n+1}=-\frac{as}{b}$$

and
$$CC'=-\frac{as}{c}\qquad\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

The circle thus determined circumscribes the three escribed circles, and its intercepts on the sides of the triangle, produced in both directions, may at once be written down, viz.

$\frac{as}{b}$, $\frac{as}{c}$ on AB and AC produced,

$\frac{bs}{c}$, $\frac{bs}{a}$ on BC and BA ,, ,

$\frac{cs}{a}$, $\frac{cs}{b}$ on CA and CB ,, .

The radius ρ of the circle is determined by (8), whence it may be shown that

$$4\left(r+\frac{r_2r_3}{r_1}\right)\rho=(r_2+r_3)(r_1-r)+\left(r+\frac{r_2r_3}{r}\right)^2,$$

therefore $$4r\rho=\frac{(b+c)(c+a)(a+b)-abc}{2s}$$

$$=\frac{(r_1+r)(r_2+r)(r_3+r)-(r_1-r)(r_2-r)(r_3-r)}{2r}$$

$$=r^2+r_2r_3+r_3r_1+r_1r_2$$

$$=r^2+\frac{r_1r_2r_3}{r}$$

$$=r^2+s^2 \quad \ldots\ldots\ldots\

of the two circles satisfying the prescribed condition; and for the other we have

$$n\left(r+\frac{r_2r_3}{r_1}\right)=r_1+r_2+r_3+\frac{r_2r_3}{r_1}$$

$$=r_1+\frac{r_2r_3}{r},$$

therefore

$$n=\frac{r_1}{r},$$

$$=\frac{s}{s-a},$$

and the position of $B'C'$ is determined.

If ρ_a be the radius of this circle, we obtain

$$4\frac{b+c}{a}r\rho_a=\frac{(b+c)(c+a)(a+b)-abc}{2s}$$

$$=4r\rho,$$

therefore $\quad \rho_a : \rho :: a : b+c :: r_1-r : r_1+r.$

Similarly the radii ρ_b, ρ_c of the two analogous circles may be written down.

6. Hence

$$\left(\frac{1}{\rho}+\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}\right)\rho=1+\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}$$

$$=\frac{(b+c)(c+a)(a+b)-abc}{abc}$$

$$=4r\rho\cdot\frac{2s}{abc}$$

$$=2\frac{\rho}{R},$$

therefore

$$\frac{1}{\rho}+\frac{1}{\rho_a}+\frac{1}{\rho_b}+\frac{1}{\rho_c}=\frac{1}{\rho'},$$

and

$$\frac{\rho_a\rho_b\rho_c}{\rho^3}=\frac{(r_1-r)(r_2-r)(r_3-r)}{(r_1+r)(r_2+r)(r_3+r)}$$

$$=\frac{\rho'}{\rho+\rho'};$$

also
$$\frac{\rho_a+\rho_b+\rho_c+\rho}{\rho}=\frac{r_1-r}{r_1+r}+\frac{r_2-r}{r_2+r}+\frac{r_3-r}{r_3+r}+1$$
$$=\frac{3\rho-r}{\rho+\rho'}.$$

If the radius of the circle which circumscribes the three escribed circles, viz. the radius determined by (11) be taken as positive; and if for each reversal of the kind of contact the sign of the radius is changed; then ρ_a, ρ_b, ρ_c will also be positive, but ρ' will be negative.

Therefore for the five circles which touch the three escribed circles we have

$$\Sigma\frac{1}{\rho}=0.$$

7. If the circle (7) passing through the points B', C', and touching the escribed circles opposite B and C, also touches the inscribed circle, whose equation is

$$(a_1-\tfrac{1}{2}r)\frac{x_1}{\rho_1}+(a_2-\tfrac{1}{2}r)\frac{x_2}{\rho_2}+(a_3-\tfrac{1}{2}r)\frac{x_3}{\rho_3}-\tfrac{1}{2}r\frac{x_4}{\rho_4}=0,$$

the following condition must be satisfied, viz.

$$\left\{1-n(n-1)\frac{r_2r_3}{\rho_1^2}\right\}(a_1-\tfrac{1}{2}r)+\left\{n+(n-1)\frac{r_2r_3}{\rho_2^2}\right\}(a_2-\tfrac{1}{2}r)$$
$$+\left\{n+(n-1)\frac{r_2r_3}{\rho_3^2}\right\}(a_3-\tfrac{1}{2}r)$$
$$-\left\{1+(n^2-1)\frac{r_2r_3}{\rho_1^2}+(n-1)\frac{r_2r_3}{\rho_4^2}\right\}.\tfrac{1}{2}r$$
$$=\pm\tfrac{1}{2}\{(r_1-r)+n^2(r_2+r_3)\}\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

If the contact is of the same character, the negative sign must be taken on the right-hand side. The terms in n^2 are found to cancel out, and there remains

$$n\left(r_1+\frac{r_2r_3}{r}\right)=r+\frac{r_2r_3}{r_1},$$

therefore
$$n=\frac{r}{r_1}.$$

The radius ρ_{bc} of this circle is given by

$$4(r_1+r)\rho_{bc}=(r_1-r)r_1+\frac{r^2}{r_1}(r_2+r_3),$$

therefore $$4\frac{(b+c)\,S}{s(s-a)}\rho_{bc}=\frac{aS^2}{s(s-a)^2}+\frac{a(s-a)\,S^2}{s^2(s-b)(s-c)},$$

therefore $$4\frac{b+c}{a}r_1\rho_{bc}=r_1^2+(s-a)^2.$$

8. If the contact with the inscribed circle be not of the same character as with the two escribed circles, the positive sign must be taken on the right of (12), whence

$$(n^2-n)(r_2+r_3)=(n-1)\left(r_1+\frac{r_2r_3}{r_1}\right),$$

therefore $$n=1,$$

or $$n(r_2+r_3)=r_1+\frac{r_2r_3}{r_1},$$

i.e. $$n+1=\frac{bc}{a(s-a)}.$$

The solution $n=1$ gives the nine-point circle, whose radius is ρ'.

The second value of n shows that the intercept BB' is $-r/r_1$ times the corresponding intercepts made by the circle which circumscribes the three escribed circles of the triangle. If ρ'' be the radius of the circle, we have

$$4\left(r_1+\frac{r_2r_3}{r}\right)\rho''=(r_2+r_3)(r_1-r)+\left(r_1+\frac{r_2r_3}{r_1}\right)^2.$$

We have already determined a radius ρ in § 4 by the equation

$$4\left(r+\frac{r_2r_3}{r_1}\right)\rho=(r_2+r_3)(r_1-r)+\left(r+\frac{r_2r_3}{r}\right)^2$$

leading to $$4r\rho=r^2+s^2.$$

If here r and r_1 be interchanged and likewise r_2 and $-r_3$ (this is equivalent more briefly to merely changing the sign of a), then we obtain the same equation as that which determines ρ''. Making the same change then in the value of ρ, we have at once

$$4r_1\rho''=r_1^2+(s-a)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (13).$$

9. We shall now prove that the circle represented by the equation (1), which passes through points $B'C'$ will, for a certain value of k, touch both the inscribed circle and the

escribed circle opposite to A, but with contacts of different character.

The condition that it should touch the inscribed circle may at once be written down. It is

$$\{(n(1+k)+n+1\}(a_1-\tfrac{1}{2}r)+\left\{n(n+1)-(1+k)\frac{\rho_1^2}{\rho_2^2}\right\}(a_2-\tfrac{1}{2}r)$$
$$+\left\{n(n+1)-(1+k)\frac{\rho_1^2}{\rho_3^2}\right\}(a_3-\tfrac{1}{2}r)$$
$$+\left\{(n+1)(1+k)-(n+1)+(1+k)\frac{\rho_1^2}{\rho_4^2}\right\}\tfrac{1}{2}r$$
$$=(n+1)\sqrt{(k^2a_1^2+n^2a^2)},$$

therefore

$$a_1-r+n(a_2+a_3-r)+\frac{1+k}{n+1}\left\{na_1-\tfrac{1}{2}\rho_1^2\left(\frac{1}{r}+\frac{1}{r_1}\right)\right\}$$
$$=\sqrt{(k^2a_1^2+n^2a^2)},$$

therefore

$$a_1-r+nr_1+\frac{1+k}{n+1}\left\{na_1-\tfrac{1}{2}\rho_1^2\left(\frac{1}{r}+\frac{1}{r_1}\right)\right\}=\sqrt{(k^2a_1^2+n^2a^2)}.$$

The condition that it should touch the circle opposite A with contact of different character is obtained similarly, or it may be written down by changing the sign of a_1 and interchanging r and r_1. It is

$$-a_1-r_1+nr+\frac{1+k}{n+1}\left\{-na_1-\tfrac{1}{2}\rho_1^2\left(\frac{1}{r}+\frac{1}{r_1}\right)\right\}=-\sqrt{(k^2a_1^2+n^2a^2)},$$

the sign of the radical being changed to express the change in the character of the contact. Thus by addition

$$(n-1)(r+r_1)-\frac{1+k}{n+1}\frac{\rho_1^2}{rr_1}(r+r_1)=0,$$

therefore
$$1+k=(n^2-1)\frac{rr_1}{\rho_1^2},$$

and
$$2ka_1=n^2(r_1-r)-(r_2+r_3),$$

therefore
$$4k^2a_1^2+4n^2a^2=\{n^2(r_1-r)+r_2+r_3)\}^2.$$

Also by subtraction

$$2a_1+(n+1)(r_1-r)+2\frac{1+k}{n+1}na_1=2\sqrt{(k^2a_1^2+n^2a^2)}.$$

With the same value of k the left-hand member becomes

$$2a_1 + (n+1)(r_1 - r) + n(n-1)(r_1 - r),$$

i.e. $$2a_1 + r_1 - r + n^2(r_1 - r),$$

therefore $$r_2 + r_3 + n^2(r_1 - r) = 2\sqrt{(k^2 a_1^2 + n^2 a^2)}.$$

Therefore the condition is satisfied and the theorem is established.

The equation to the circle in question is found by substituting the corresponding value of k. It is

$$\left\{1 + n(n-1)\frac{rr_1}{\rho_1^2}\right\}\rho_1 x_1 + \left\{n - (n-1)\frac{rr_1}{\rho_2^2}\right\}\rho_2 x_2 + \left\{n - (n-1)\frac{rr_1}{\rho_3^2}\right\}\rho_3 x_3$$
$$+ \left\{1 - (n^2-1)\frac{rr_1}{\rho_1^2} - (n-1)\frac{rr_1}{\rho_4^2}\right\}\rho_4 x_4 = 0 \quad \ldots\ldots(14).$$

From (2) it follows that the radius ρ is given by

$$4(n+1)\rho = n^2(r_1 - r) + r_2 + r_3 \quad \ldots\ldots(15).$$

10. We now proceed to determine the value of n, and the radius, in order that this circle may also touch the circle escribed opposite to B. The condition to be satisfied is

$$\left\{1 + n(n-1)\frac{rr_1}{\rho_1^2}\right\}(a_1 - \tfrac{1}{2}r_3) - \left\{n - (n-1)\frac{rr_1}{\rho_2^2}\right\}(a_2 + \tfrac{1}{2}r_3)$$
$$+ \left\{n - (n-1)\frac{rr_1}{\rho_3^2}\right\}(a_3 - \tfrac{1}{2}r_3)$$
$$- \left\{1 - (n^2-1)\frac{rr_1}{\rho_1^2} - (n-1)\frac{rr_1}{\rho_4^2}\right\}\tfrac{1}{2}r_3 = \pm\tfrac{1}{2}\{r_2 + r_3 + n^2(r_1 - r)\},$$

the sign on the right depending on the nature of the contact. Therefore

$$a_1 - r_2 + \tfrac{1}{2}\frac{rr_1}{r_2 r_3}(r_3 - r_2) - n\left(\frac{rr_1}{r_2} + r_3\right) + \tfrac{1}{2}n^2(r_1 - r)$$
$$= \pm\tfrac{1}{2}\{r_2 + r_3 + n^2(r_1 - r)\},$$

therefore

$$\tfrac{1}{2}(r_3 - r_2) - \tfrac{1}{2}(r_1 - r) + \tfrac{1}{2}\frac{rr_1}{r_2 r_3}(r_3 - r_2) - n\left(\frac{rr_1}{r_2} + r_3\right) + \tfrac{1}{2}n^2(r_1 - r)$$
$$= \pm\tfrac{1}{2}\{r_2 + r_3 + n^2(r_1 - r)\},$$

therefore

$$\tfrac{1}{2}\{r_2+r_3+n^2(r_1-r)\}-r_2-\frac{rr_1}{r_3}-n\left(\frac{rr_1}{r_2}+r_3\right)$$
$$=\pm\tfrac{1}{2}\{r_2+r_3+n^2(r_1-r)\}.$$

If we take the positive sign on the right-hand side, the contact with the two escribed circles will be of different characters. We then have

$$n=-\frac{r_2}{r_3},$$

and $$4(n+1)\rho=r_2+r_3+n^2(r_1-r),$$

therefore $$4(r_3\sim r_2)\rho=(r_2+r_3)r_3+\frac{r_2^2}{r_3}(r_1-r),$$

therefore $$4\frac{c\sim b}{s-c}.S.\rho=\frac{aS^2}{(s-c)^2}+a(s-c)^2,$$

therefore $$\frac{4(c\sim b)}{a}r_3\rho=r_3^2+(s-c)^2 \quad\ldots\ldots\ldots\ldots(16).$$

If we take the sign on the right-hand side to be negative, the contacts with the two escribed circles will be of the same character, and we have

$$n^2(r_1-r)-n\left(r_1-r-\frac{rr_1}{r_3}+r_3\right)-\frac{rr_1}{r_3}+r_3=0,$$

therefore $$(n^2-n)(r_1-r)+(n-1)\left(\frac{rr_1}{r_3}-r_3\right)=0,$$

therefore $$n=1$$

or $$n=\left(r_3-\frac{rr_1}{r_3}\right)\Big/(r_1-r),$$

i.e. $$n+1=\left(r_3+\frac{rr_1}{r_3}\right)\Big/(r_1-r)$$
$$=\frac{bc}{a(s-c)} \quad\ldots\ldots\ldots\ldots(17).$$

The solution $n=1$ gives the nine-point circle, whose radius $\rho'=\frac{1}{2}R$.

The remaining solution gives for the determination of the corresponding radius ρ'' the equation

$$4\left(r_3+\frac{rr_1}{r_3}\right)\rho''=\left(r_3-\frac{rr_1}{r_3}\right)^2+(r_2+r_3)(r_1-r).$$

Comparing this with the equation to determine ρ in § 4, viz.

$$4\left(r+\frac{r_3r_3}{r_1}\right)\rho=\left(r+\frac{r_3r_3}{r}\right)^2+(r_2+r_3)(r_1-r),$$

it is obvious that if in the latter we interchange r and r_3, and also r_1 and $-r_3$, or briefly if we change the sign of c, we obtain the former. Therefore since

$$4r\rho=r^2+s^2$$

we have $$4r_3\rho''=r_3^2+(s-c)^2.$$

This is obviously one of the three circles of the type already investigated from a different aspect in § 8, viz. the circles, other than the nine-point circle, which touch two of the escribed circles similarly and the inscribed circle differently.

11. We have thus determined all the five circles which touch any two escribed circles (say on b, c) of the triangle and also the inscribed circle, viz.

(1) Touching the three circles externally

$$-4\frac{b+c}{a}r_1\rho=r_1^2+(s-a)^2,$$

the minus sign is prefixed, the radius being here assumed to be negative.

(2) Two circles touching the two escribed circles externally and the inscribed circle internally, viz. the nine-point circle and another, the radii being given by

$$\rho'=\tfrac{1}{2}R$$

and $$4r_1\rho''=r_1^2+(s-a)^2,$$

these two radii being positive, since one contact has been reversed in comparing with (1).

(3) Two circles touching the inscribed and one of the escribed circles internally (or externally), and the other escribed circle externally (or internally), the radii being given by

$$-4\frac{a-b}{c}r_1\rho=r_1^2+(s-a)^2,$$

and $$-4\frac{a-c}{b}r_1\rho=r_1^2+(s-a)^2.$$

It will be found that the right sign is here given for the contacts which occur. For instance it appears that if a is greater than both b and c, the contact with the inscribed circle is internal, and there is one reversal of contact in comparison with (2).

Now all these results are identical with those for the five circles touching the three escribed circles if the sign of a be changed in all of them, and the consequent changes in s, r, r_1, r_2, r_3 be made. Analytically it is the same problem over again with the sign of a changed throughout, whether occurring explicitly or implicitly.

12. There are thus in general four groups of five circles determined, the nine-point circle being common to all the groups; so that the total number of circles touching three of the four circles escribed or inscribed in a triangle is in general seventeen. If the triangle is equilateral there are only eight distinct circles, for six of the seventeen degenerate into straight lines, and three others coincide with the nine-point circle.

13. We may now give a very simple geometrical construction to determine the points of intersection of the sides of the triangle with the circle which touches the inscribed circle internally and two of the escribed circles externally.

Let the two escribed circles to be so touched be those opposite to the angular points B, C of the triangle.

Let the points of intersection be

B', C'; $B'C'$ being parallel to BC,

C'', A''; $C''A''$ ... CA,

A''', B'''; $A'''B'''$ ... AB.

Then it has been proved in § 9 that

$$BB' = \frac{a(s-a)}{b}, \quad CC' = \frac{a(s-a)}{c};$$

also from (17), *mutatis mutandis*, it appears that

$$CC'' = \frac{b(s-a)}{c}, \quad AA'' = \frac{b(s-a)}{a},$$

$$AA''' = \frac{c(s-a)}{a}, \quad BB''' = \frac{c(s-a)}{b}.$$

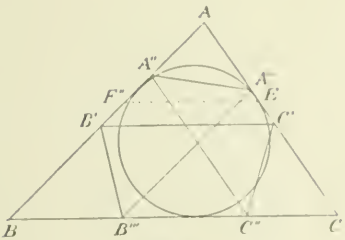

Fig. 2.

In fig. 2 let E be the point of contact of AC with the nine-point circle.

Draw EF'' parallel to CB, to meet AB in F''. Measure

$$BB' = EF'' = \frac{a(s-a)}{b}.$$

Draw $B'B'''$ meeting BC at B''', so that the angle

$$B'B'''B = A,$$

then
$$BB''' = \frac{c(s-a)}{b}.$$

Draw $B'''A'''$ meeting AC in A''',

then
$$AA''' = \frac{c(s-a)}{a}.$$

Draw $A'''A''$ meeting AB in A'', so that the angle

$$A'''A''A = C,$$

then
$$AA'' = \frac{b(s-a)}{a}.$$

Draw $A''C''$ parallel to AC, to meet BC in C'', then

$$CC'' = \frac{b(s-a)}{c}.$$

Also $B'C'$ is parallel to BC,

therefore
$$CC' = \frac{b}{c} BB' = \frac{a(s-a)}{c}.$$

Therefore
$$CC'' : CC' :: b : a.$$

Therefore the angle
$$C'C''C = A$$

and
$$CC'C'' = B.$$

The six points A', B', B'', C'', C''', A''' are the required points of intersection of the circle with the sides of the triangle.

This circle is a Tucker circle of the triangle, and it touches the inscribed and the two escribed circles.

If K, O_1 be the symmedian point of the triangle and the centre of the Lemoine circle respectively, the centre, O_2, of this Tucker circle lies on KO_1 and

$$OO_2 : OO_1 :: BB' : BF'$$
$$:: (a^2+b^2+c^2)(s-a) : abc.$$

There are clearly two other analogous Tucker circles.

A similar construction serves for determining the points of intersection of the circle which circumscribes the three escribed circles (*vide* § 4), an escribed circle being used instead of the inscribed circle here employed.

The centres of these four circles all lie on the line KO.

For the three of these circles which touch the inscribed circles the intercepts BB', measured towards A, have been found to be

$$\frac{a}{b}(s-a), \quad \frac{a}{b}(s-b), \quad \frac{a}{b}(s-c),$$

and their sum is as/b.

But this is the corresponding intercept BB' made in the reverse direction by the circle which touches the three escribed circles, but not the inscribed. This gives another geometrical determination of the six intercepts made by that circle.

14. Equation (7) is that of the circle which passes through B', C', and touches the two circles escribed opposite B and C.

The other point of its intersection with AB or AC may be determined by writing

$$\rho_4 x_4 = \rho_3 x_3 \text{ or } \rho_2 x_2.$$

Take
$$\rho_4 x_4 = \rho_3 x_3.$$

The equation becomes

$$\left\{1-n(n-1)\frac{r_2 r_3}{\rho_1^2}\right\}\rho_1 x_1 + \left\{n+(n-1)\frac{r_3 r_1}{\rho_2^2}\right\}\rho_2 x_2$$
$$+\left\{n+1-(n-1)\,r_2 r_3 \frac{c^2}{\rho_1^2 \rho_2^2} + (n^2-1)\frac{r_2 r_3}{\rho_1^2}\right\}\rho_3 x_3 = 0.$$

But at the same time

$$\frac{x_1}{\rho_1} + \frac{x_2}{\rho_2} - \frac{c^2}{\rho_1^2 \rho_2^2}\rho_3 x_3 = -2;$$

therefore, substituting the value thus obtained for $\rho_3 x_3$, we have

$$\left\{1-(n^2-1)\frac{r_2r_3}{c^2}+(n+1)\frac{\rho_2^2}{c^2}\right\}\rho_1x_1$$
$$+\left\{n+(n^2-1)\frac{r_3r_1}{c^2}+(n+1)\frac{\rho_1^2}{c^2}\right\}\rho_2x_2+2(n+1)\frac{\rho_1^2\rho_2^2}{c^4}$$
$$-2(n-1)r_2r_3+2(n^2-1)r_2r_3\frac{\rho_2^2}{c^2}=0 \quad \ldots\ldots\ldots\ldots (18).$$

This is known to be satisfied at the point B', where

$$\rho_1x_1+\rho_1^2-\frac{n^2c^2}{(n+1)^2}=0,$$

and

$$\rho_2x_2+\rho_2^2-\frac{c^2}{(n+1)^2}=0.$$

Let $\rho_1x_1+\rho_1^2$ be ξ^2, and $\rho_2x_2+\rho_2^2=\eta^2$.

Thus the equation (18) must reduce to

$$\left\{\xi^2-\frac{n^2c^2}{(n+1)^2}\right\}\left\{1-(n^2-1)\frac{r_2r_3}{c^2}+(n+1)\frac{\rho_2^2}{c^2}\right\}$$
$$+\left\{\eta^2-\frac{c^2}{(n+1)^2}\right\}\left\{n+(n^2-1)\frac{r_3r_1}{c^2}+(n+1)\frac{\rho_1^2}{c^2}\right\}=0.$$

But if

$$A(\xi^2-p^2)+B(\eta^2-q^2)=0,$$

and

$$\xi+\eta=p+q,$$

then, in addition to $\xi=p$, there is the solution

$$(A+B)\xi+(A-B)p-2Bq=0.$$

This gives for the second intersection of the circle (7) with AB

$$c\xi=\rho_1^2+(n-1)r_2r_3$$
$$=nr_2r_3-rr_1.$$

Here ξ is the distance of the point of intersection from A, reckoned positive in the direction AB. If the distances, ξ', be measured from the foot of the perpendicular dropped from C on AB,

$$\xi'=\xi-\frac{\rho_1^2}{c}=(n-1)\frac{r_2r_3}{c}.$$

For the corresponding distances along AC

$$\xi'' = (n-1)\frac{r_2 r_3}{b}.$$

Moreover equation (14), which is that of a circle touching the in-circle and the escribed circle opposite A, may be derived from (7) by changing the sign of either b or c. For changing the sign of b

r becomes r_2,

and r_1 ,, $-r_3$,

and *vice versa;* which converts either equation into the other.

The distances ξ', ξ'' in this case are therefore given by

$$\xi' = -(n-1)\frac{rr_1}{c},$$

$$\xi'' = -(n-1)\frac{rr_1}{b}.$$

Now all the circles which touch any three of the four circles inscribed or escribed to the triangle have been shown to be of the type represented by equation (7) or (14). Thus in every case at least four points are determined by employing the appropriate value of n already found. Nine such circles have with the triangle a common chord parallel to each side. Five of them, viz. the nine-point circle and the four Tucker circles of § 13, are common to each set of nine. The remaining four are in general distinct for the three sets. The four that pass through points B', C' are

(1) in § 5, where $n = \frac{r_1}{r}$

and $\xi_1' = \frac{as}{c}$,

(2) in § 7, where $n = \frac{r}{r_1}$

and $\xi_2' = -\frac{a(s-a)}{c}$,

(3) in § 10, where $n = -\frac{r_2}{r_3}$

and $\xi_3' = \frac{a(s-c)}{c}$,

(4) and similarly where $n = -\frac{r_3}{r_2}$

and $$\xi_4' = \frac{a(s-b)}{c}.$$

These intercepts on the side c are the same in magnitude as those made by the four Tucker circles on the side b, measured from the point C, which have already been determined by an easy geometrical method in fig. 2. This result is remarkable, and enables all the required points of intersection to be obtained for all the tangent circles under consideration.

15. The general question of Tucker circles arises out of this investigation.

The equation (1) to a circle passing through points B', C', where $B'C'$ is parallel to BC, may be written in the following form:

$$(1+k)\{(n+1)(\rho_1 x_1 - \rho_4 x_4) + 2\rho_1^2\}$$
$$+ (n+1)\{\rho_1 x_1 + \rho_4 x_4 + n(\rho_2 x_2 + \rho_3 x_3)\} = 0;$$

therefore when $\rho_1 x_1 = \rho_4 x_4$, *i.e.* at the points where it cuts BC,

$$(1+k)\,2\rho_1^2 + (n+1)\{2\rho_1 x_1 + n(\rho_2 x_2 + \rho_3 x_3)\} = 0 \ldots (19).$$

If the circle passes through B'' (fig. 2), for any value of n, *i.e.* for any position of the point B' on AB, where

$$BB'' : BB' :: c : a,$$

then it will be a Tucker circle and pass through C'', A'', A'''. At B''

$$\rho_2 x_2 + \rho_3^2 = \frac{c^4}{(n+1)^2 a^2},$$

and $$\rho_2 x_2 + \rho_3^2 = \left\{a - \frac{c^2}{(n+1)a}\right\}^2,$$

therefore $$\rho_2 x_2 + \rho_3 x_3 = \frac{2c^4}{(n+1)^2 a^2} - \frac{2c^2}{n+1},$$

Also, since $$\Sigma \frac{x_k}{\rho_k} = -2,$$

we obtain $$2\rho_1 x_1 = 2\rho_2^2 - \frac{4\rho_2^2 c^2}{(n+1)a^2} + \frac{2c^4}{(n+1)^2 a^2},$$

whence $2\rho_1 x_1 + n(\rho_2 x_2 + \rho_3 x_3) = -2\rho_1^2 + \dfrac{2b^2c^2}{(n+1)a^2}$.

Then by substitution in (19) it follows that for a Tucker circle the value of k is given by

$$1 + k = n + 1 - \frac{b^2c^2}{a^2\rho_1^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

The equation to the Tucker circle is therefore

$$\left(n + 1 - \frac{b^2c^2}{a^2\rho_1^2}\right)\{(n+1)(\rho_1 x_1 - \rho_4 x_4) + 2\rho_1^2\} + (n+1)\{\rho_1 x_1 + \rho_4 x_4 + n(\rho_2 x_2 + \rho_3 x_3)\} = 0,$$

which may be written

$$(n+1)^2(\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4) + (n+1)\{\rho_1 x_1 - \rho_2 x_2 - \rho_3 x_3 + \rho_4 x_4 - \frac{b^2c^2}{a^2\rho_1^2}(\rho_1 x_1 - \rho_4 x_4) + 2\rho_1^2\} - \frac{2b^2c^2}{a^2} = 0;$$

and this reduces to

$$(n+1)\left(n + 1 - \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2}\right)(\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4) + \frac{n+1}{a^2}(a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3) + 2(n+1)\frac{b^2c^2 - \rho_1^4}{a^2} - \frac{2b^2c^2}{a^2} = 0.$$

This becomes

$$(n+1)\left(n + 1 - \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2}\right)(\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4) + \frac{n+1}{a^2}\left(a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3 + \frac{2\rho_1^2\rho_2^2\rho_3^2}{\rho_1^2 + \rho_2^2 + \rho_3^2}\right) + \left(n + 1 - \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2}\right)\frac{2b^2c^2}{\rho_1^2 + \rho_2^2 + \rho_3^2} = 0 \ldots (21).$$

These equations may at once be rendered homogeneous by means of the relation

$$\Sigma \frac{x_k}{\rho_k} = -2.$$

16. Since

$$\rho_1 x_1 + \rho_2 x_2 + \rho_3 x_3 - \rho_4 x_4 = 0$$

is the equation to the circumcircle, we will write it

$$C_1 = 0;$$

this circle is seen to be the Tucker circle for which $n = \infty$.

If $$n + 1 = \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2},$$

BB' is twice the length that it has for Lemoine's circle, which is another Tucker circle.

With this value of $n+1$, equation (21) reduces to

$$a^2\rho_1 x_1 + b^2\rho_2 x_2 + c^2\rho_3 x_3 + \frac{2\rho_1^2\rho_2^2\rho_3^2}{\rho_1^2 + \rho_2^2 + \rho_3^2} = 0 \quad \ldots\ldots(22).$$

This then is the Tucker circle which has its centre at K, and if we write it $K_1 = 0$, the general equation including all Tucker circles, whatever the value of n, may be written

$$(n+1)\left(n - \frac{\rho_1^2}{a^2}\right) C_1 + \frac{n+1}{a^2} K_1 + \left(n - \frac{\rho_1^2}{a^2}\right) \frac{2b^2c^2}{\rho_1^2 + \rho_2^2 + \rho_3^2} = 0.$$

The centre obviously lies on the line joining the centres of C_1 and K_1, *i.e.* O and K.

The equation to Lemoine's circle, for which

$$n + 1 = \frac{2(\rho_1^2 + \rho_2^2 + \rho_3^2)}{a^2},$$

is $$C_1 + \frac{1}{\rho_1^2 + \rho_2^2 + \rho_3^2} K_1 + \frac{a^2b^2c^2}{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2} = 0 \quad \ldots\ldots(23),$$

which agrees with the result found in the *Messenger*, vol. xlv., p. 91.

17. The radius of a Tucker circle is given by (2), viz.

$$4(n+1)^2\rho^2 = k^2a_1^2 + n^2a^2,$$

where k has the value given by (20), viz.

$$k = n - \frac{b^2c^2}{a^2\rho_1^2}.$$

Therefore

$$4(n+1)^2\rho^2 = 4n^2R^2 - 2n\frac{b^2c^2a_1^2}{a^2\rho_1^2} + \frac{b^4c^4a_1^2}{a^4\rho_1^4}.$$

But $$b^2c^2a_1^2 = 4R^2\rho_1^4,$$

therefore

$$(n+1)^2\rho^2 = R^2\left(n^2 - 2n\frac{\rho_1^2}{a^2} + \frac{b^2c^2}{a^4}\right)$$

$$= R^2\left\{\left(n - \frac{\rho_1^2}{a^2}\right)^2 - \frac{\rho_1^2\rho_2^2\rho_3^2}{a^4\rho_4^2}\right\} \ldots\ldots (24).$$

Hence for the circle K_4, the centre of which coincides with the symmedian point of the triangle,

$$\rho = 2k'R,$$

where the distance of the centre K from any side a is $k'a$. For this circle $B'C'$ and BC are equidistant from k, and similarly for the other sides.

For Lemoine's circle,

$$n + 1 = 2\frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2},$$

therefore

$$4(\rho_1^2 + \rho_2^2 + \rho_3^2)^2\rho^2 = R^2\left\{(\rho_1^2 + \rho_2^2 + \rho_3^2)^2 - \frac{\rho_1^2\rho_2^2\rho_3^2}{\rho_4^2}\right\},$$

therefore $$4\rho^2 = R^2(1 + 4k'^2).$$

Equation (24) may also be written

$$\rho^2 = R^2\left[\frac{\{(n+1)a^2 - (1+4k'^2)(\rho_1^2 + \rho_2^2 + \rho_3^2)\}^2}{(n+1)^2(1+4k'^2)a^4} + \frac{4k'^2}{1+4k'^2}\right].$$

Therefore the smallest Tucker circle is that for which

$$n + 1 = (1 + 4k'^2)\left(1 + \frac{\rho_1^2}{a^2}\right),$$

and its radius is given by

$$\rho^2 = \frac{4k'^2}{1 + 4k'^2}R^2.$$

The radii of the Tucker circles which touch any three of the circles in- or e-scribed to a triangle have already been determined.

18. The condition that a Tucker circle may touch the inscribed circle is, as in § 9,

$$a_1 - r - r_1 + (n+1)r_1 + \frac{1+k}{n+1}\left\{na_1 - \tfrac{1}{2}\rho_1^2\left(\frac{1}{r} + \frac{1}{r_1}\right)\right\} = \pm\sqrt{(k^2a_1^2 + n^2a^2)},$$

k being given the appropriate value for a Tucker circle, viz.

$$1 + k = n + 1 - \frac{b^2c^2}{a^2\rho_1^2}.$$

This becomes

$$\left[(n+1)^2r_1 - (n+1)(r_1 + r - a_1) + \left(n + 1 - \frac{b^2c^2}{a^2\rho_1^2}\right)\left\{(n+1)a_1 - \frac{\rho_1^2}{r}\right\}\right]^2$$

$$= 4(n+1)^2R^2\left\{(n+1 - \frac{\rho_1^2 + \rho_2^2 + \rho_3^2}{a^2}\right)^2 - \frac{\rho_1^2\rho_2^2\rho_3^2}{a^4\rho_4^2}\Bigg\},$$

therefore $(n+1)^4\{(a_1 + r_1)^2 - 4R^2\}$ + terms in the first, second and third powers of $n+1$

$$+ \frac{b^4c^4}{a^4r^2} = 0.$$

Therefore the products of the four values of $n+1$ for which the corresponding Tucker circles touch the inscribed circle is

$$\frac{4b^4c^4}{a^4r^2} \div \{(r_1 + r_2 + r_3 + r)^2 - (r_1 + r_2 + r_3 - r)^2\},$$

i.e.

$$\frac{b^4c^4}{a^4r^3(r_1 + r_2 + r_3)}.$$

But the value of $n+1$ for three of these circles has already been determined, viz., $bc/a(s-a)$, $bc/a(s-b)$, $bc/a(s-c)$; and the product of these is

$$\frac{b^3c^3}{a^3(s-a)(s-b)(s-c)}.$$

There is then a fourth Tucker circle which touches the inscribed circle, and it is determined by

$$n + 1 = \frac{bc(s-a)(s-b)(s-c)}{ar^3(r_1 + r_2 + r_3)} = \frac{4s^2R}{a^2(4R + r)}.$$

Similarly in addition to the three already determined there is a fourth Tucker circle touching each of the escribed circles. For the escribed circle opposite A the corresponding value of $n+1$ is $4(s-a)^2R/a^2(4R - r_1)$; and similarly for the two others.

PARAMETRIC EXPRESSIONS FOR THE COORDINATES OF ANY POINT ON A SURFACE NEAR AN ORIGIN.

By *Professor A. R. Forsyth.*

1. It is a known property of any skew curve that, given its circular curvature and its tortuosity, the curve is uniquely determinate save as to its position and orientation in space. When any point on the curve is taken as origin, and the principal lines at that origin are chosen as axes of reference, the coordinates of any point on the curve can be obtained in series of powers of s, the arc-distance of the point from the origin.*

It is an equally known property of any surface that, given its fundamental magnitudes of the first order and the second order obeying the Gauss equation and the two Mainardi-Codazzi equations, the surface is uniquely determinate save as to its position and orientation in space. When the lines of curvature are taken as the curves of reference for the surface, the Gauss equation is

$$LN = -\tfrac{1}{2}(E_{22} + G_{11}) + \frac{1}{4G}(E_2 G_2 + G_1^2) + \frac{1}{4E}(E_1 G_1 + G_2^2);$$

and the Mainardi-Cadazzi equations are

$$L_2 = \tfrac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) E_2, \quad N_1 = \tfrac{1}{2}\left(\frac{L}{E} + \frac{N}{G}\right) G_1,$$

the quantities L/E and N/G being the principal curvatures.

2. Now take some point on the surface as an origin; take also the normal and the tangent lines to the lines of curvature through that origin as axes of reference: then, by the customary theory, we have

$$2z = \frac{L}{E} x^2 + \frac{N}{G} y^2 + \text{higher powers},$$

as the equation of the surface in the immediate vicinity of the origin.

But another form can be given to the analytical representation of the surface by expressing the coordinates of any

* G. B. Mathews, *Quart. Journ*, vol. xxvi. (1893): the expressions up to the ninth order of small quantities are given by S. Mukhopadhyay, *Bull. Calc. Math. Soc.* (1915).

point, referred to the foregoing axes, in terms of two surface-parameters. Denote the origin by O, and the lines of curvature at O by OM and ON; through any other point P draw the lines of curvature meeting the lines OM and ON in M and N; the position of P is uniquely determined by s, the length of OM, and t, the length of ON. The parameters of the lines OM, ON, PN, PM, and respectively constant along them, will be denoted by q_0, p_0, q, p; to shorten printing, which would involve $p-p_0$, $q-q_0$, we shall suppose $p_0=0$, $q_0=0$. Then the coordinates of P referred to the assumed axes at O are obtainable in series of powers of s and t, or in series of powers of p and q. I have calculated these quantities as far as the fourth order of small quantities inclusive for x and y, and as far as the third order for z; and, as I have not seen them elsewhere, it seems worth while placing the expressions on record.

3. The respective relations between s and p can be obtained by direct integration. We have (in the usual notation) $F=0$, $M=0$ everywhere, with the assumed lines of reference; so writing E_0, G_0, L_0, N_0 for the values of E, G, L, N at the origin, and denoting derivatives $\partial E/\partial p$, $\partial E/\partial q$, $\partial^2 E/\partial p^2$, ..., at O, by E_1, E_2, E_{11}, ..., respectively, we have

$$ds^2 = E(p, 0)\, dp^2,$$

and therefore

$$\begin{aligned} s &= \int_0^p \{E(p, 0)\}^{\frac{1}{2}} dp \\ &= \int_0^p E_0^{\frac{1}{2}} \left\{1 + p\,\frac{E_1}{E_0} + \tfrac{1}{2}p^2\,\frac{E_{11}}{E_0} + \tfrac{1}{6}p^3\,\frac{E_{111}}{E_0} + \ldots\right\}^{\frac{1}{2}} dp \\ &= E_0^{\frac{1}{2}}\, p + \tfrac{1}{4}\,\frac{E_1}{E_0^{\frac{1}{2}}}\,p^2 + \tfrac{1}{24}\left(2\,\frac{E_{11}}{E_0^{\frac{1}{2}}} - \frac{E_1^2}{E_0^{\frac{3}{2}}}\right)p^3 \\ &\qquad + \tfrac{1}{192}\left(4\,\frac{E_{111}}{E_0^{\frac{1}{2}}} - 6\,\frac{E_1 E_{11}}{E_0^{\frac{3}{2}}} + 3\,\frac{E_1^3}{E_0^{\frac{5}{2}}}\right)p^4 + \ldots. \end{aligned}$$

Similarly

$$\begin{aligned} t &= G_0^{\frac{1}{2}} q + \tfrac{1}{4}\,\frac{G_2}{G_0^{\frac{1}{2}}}\,q^2 + \tfrac{1}{24}\left(2\,\frac{G_{22}}{G_0^{\frac{1}{2}}} - \frac{G_2^2}{G_0^{\frac{3}{2}}}\right)q^3 \\ &\qquad + \tfrac{1}{192}\left(4\,\frac{G_{222}}{G_0^{\frac{1}{2}}} - 6\,\frac{G_2 G_{22}}{G_0^{\frac{3}{2}}} + 3\,\frac{G_2^3}{G_0^{\frac{5}{2}}}\right)q^4 + \ldots. \end{aligned}$$

When these series are inverted, so as to express p in terms of s and q in terms of t, we find—up to the same order—

$$p = \frac{1}{E_0^{\frac{1}{2}}} s - \tfrac{1}{4} \frac{E_1}{E_0^2} s^2 + \tfrac{1}{12} \frac{1}{E_0^{\frac{3}{2}}} (2E_1^2 - E_0 E_{11}) s^3 - \tfrac{1}{96} \frac{1}{E_0^5} (2E_{111} E_0^2 - 13 E_{11} E_1 E_0 + 14 E_1^3) s^4 + \dots,$$

$$q = \frac{1}{G_0^{\frac{1}{2}}} t - \tfrac{1}{4} \frac{G_2}{G_0^2} t^2 + \tfrac{1}{12} \frac{1}{G_0^{\frac{3}{2}}} (2G_2^2 - G_0 G_{22}) t^3 - \tfrac{1}{96} \frac{1}{G_0^5} (2G_{222} G_0^2 - 13 G_{22} G_2 G_0 + 14 G_2^3) t^4 + \dots.$$

It is clear that, to the first order of small quantities,

$$x = s, \quad y = t, \quad z = 0,$$

z itself being of the second order in its lowest terms. Further, in proceeding to approximations to any order in powers of s and t (or in powers of p and q), we take approximations for x and y to one power higher than the approximation for z.

4. Two obvious methods of obtaining the approximation present themselves.

By one of these methods, we begin with the three sets of equations—one of which is

$$\left.\begin{aligned} x_{11} &= LX + \frac{E_1}{2E} x_1 - \frac{E_2}{2G} x_2 \\ x_{12} &= \qquad \frac{E_2}{2E} x_1 + \frac{G_1}{2G} x_2 \\ x_{22} &= NX - \frac{G_1}{2E} x_1 + \frac{G_2}{2G} x_2 \end{aligned}\right\}.$$

Taking account of the fact that, in these equations, the quantities E, G, their derivatives, and L, N, X, Y, Z, x_1, x_2, y_1, y_2, z_1, z_2, are the values at the point x, y, z, we shall obtain the approximations for x, y, z up to the second order inclusive, by assigning to all these quantities their values at the origin of reference. Thus if, to this order, we write

$$\left.\begin{aligned} x &= E_0^{\frac{1}{2}} p + \tfrac{1}{2}(ap^2 + 2hpq + bq^2) \\ y &= G_0^{\frac{1}{2}} q + \tfrac{1}{2}(a'p^2 + 2h'pq + b'q^2) \\ z &= \qquad \tfrac{1}{2}(a''p^2 + 2h''pq + b''q^2) \end{aligned}\right\},$$

then (as $X=0$, $Y=0$, $Z=1$ at the origin) we have

$$\left.\begin{aligned} a &= \frac{E_1}{2E_0^{\frac{1}{2}}}, & h &= \frac{E_2}{2E_0^{\frac{1}{2}}}, & b &= -\frac{G_1}{2E_0^{\frac{1}{2}}} \\ a' &= -\frac{E_2}{2G_0^{\frac{1}{2}}}, & h' &= \frac{G_1}{2G_0^{\frac{1}{2}}}, & b' &= \frac{G_2}{2G_0^{\frac{1}{2}}} \\ a'' &= L_0, & h'' &= 0, & b'' &= N_0 \end{aligned}\right\}.$$

When we require a higher order of approximation, we either use the equations* for x_{111}, ..., together with the derived magnitudes of the third order, or we differentiate the foregoing equations for x_{11}, ...

But here an elaboration of symbols enters, connected entirely with the fundamental magnitudes. There are two magnitudes, E and G. The two magnitudes L and N are such that their product is expressible in terms of E and G and their derivatives of the second order. By taking the first derivatives of the Gauss equation with respect to p and q, and by using the Mainardi-Codazzi relations, we can express L_1, L_2, N_1, N_2 in terms of L and N (subject to the proper value of LN), and of E and G together with their derivatives up to the third order inclusive. When we proceed to the next higher order by this method, we should need to express L_{11}, L_{12}, L_{22}, N_{11}, N_{12}, N_{22} in terms of L and N, and of E and G together with their derivatives up to the fourth order inclusive. The manipulation requires a perpetually recurring reference to the equations of condition.

5. By the other method of proceeding, we still regard E, G, L, N as known, with $F=0$, $M=0$: we take expansions of x, y, z in powers of p and q with unknown coefficients and from them construct the expressions for E, F, G, L, M, N, which must be equal to the expanded values of E, 0, G, L, 0, N. When the coefficients of powers of p and q are equated, relations ensue. These relations determine the unknown coefficients. But they take us further; for they determine, in the third order of small quantities, three necessary equations through values of the coefficients and, in the fourth order of small quantities in x and y, two further necessary equations. The former three equations are the Gauss equation and the two Mainardi-Codazzi equations; the latter two equations are satisfied in virtue of the Gauss equation and its two first derivatives. The method thus gives an independent estab-

* See my *Lectures on Differential Geometry*, § 43.

lishment of the Gauss equation and the two Mainardi-Codazzi equations; save for verification in higher stages, it does not require recurring reference to these equations; and its process allows (through considerations of symmetry) a certain amount of checking of the accuracy of the calculations. Accordingly, I adopted the latter method; but it is rather laborious.

6. We take

$$\begin{aligned}
x &= E_0^{\frac{1}{2}} p + \tfrac{1}{2}(ap^2 + 2hpq + bq^2) + \tfrac{1}{6}(Ap^3 + 3Bp^2q + 3Cpq^2 + Dq^3) \\
&\quad + \tfrac{1}{24}(Pp^4 + 4Qp^3q + 6Rp^2q^2 + 4Spq^3 + Tq^4) + \dots, \\
y &= G_0^{\frac{1}{2}} q + \tfrac{1}{2}(a'p^2 + 2h'pq + b'q^2) \\
&\quad + \tfrac{1}{6}(A'p^3 + 3B'p^2q + 3C'pq^2 + D'q^3) \\
&\quad + \tfrac{1}{24}(P'p^4 + 4Q'p^3q + 6R'p^2q^2 + 4S'pq^3 + T'q^4) + \dots, \\
z &= \tfrac{1}{2}(a''p^2 + 2h''pq + b''q^2) \\
&\quad + \tfrac{1}{6}(A''p^3 + 3B''p^2q + 3C''pq^2 + D''p^3) + \dots;
\end{aligned}$$

and, merely to speed up the calculations, we shall assume the foregoing (§ 4) values of a, h, b, a', h', b', a'', h'', b'' as known. They will be found to be verified at various stages.

We have

$$\begin{aligned}
&E_0 + E_1 p + E_2 q + \tfrac{1}{2}(E_{11}p^2 + 2E_{12}pq + E_{22}q^2) \\
&\qquad + \tfrac{1}{6}(E_{111}p^3 + 3E_{112}p^2q + 3E_{122}pq^2 + E_{222}q^3 + \dots \\
&= E(p, q) \\
&= x_1^2 + y_1^2 + z_1^2.
\end{aligned}$$

When the values of x_1, y_1, z_1 are substituted, and coefficients of various powers of p and q are equated on the two sides, the terms in p and q have equal coefficients; and, for the rest, we have

$$\left.\begin{aligned}
E_0^{\frac{1}{2}} A + a^2 + a'^2 + a''^2 &= \tfrac{1}{2}E_{11} \\
2E_0^{\frac{1}{2}} B + 2ah + 2a'h' &= E_{12} \\
E_0^{\frac{1}{2}} C + h^2 + h'^2 &= \tfrac{1}{2}E_{22}
\end{aligned}\right\},$$

$$\left.\begin{aligned}
E_0^{\frac{1}{2}} P + 3aA + 3a'A' + a''A'' &= \tfrac{1}{2}E_{111} \\
E_0^{\frac{1}{2}} Q + 2aB + hA + 2a'B' + h'A' + 2a''B'' &= \tfrac{1}{2}E_{112} \\
E_0^{\frac{1}{2}} R + aC + 2hB + a'C' + 2h'B' + a''C'' &= \tfrac{1}{2}E_{122} \\
E_0^{\frac{1}{2}} S + 3hC + 3h'C' &= 2E_{222}
\end{aligned}\right\}.$$

Proceeding similarly from

$$G(p, q) = x_2^2 + y_2^2 + z_2^2,$$

to the same order of small quantities, we have

$$\left.\begin{aligned} G_0^{\frac{1}{2}} B' + h^2 + h'^2 &= \tfrac{1}{2} G_{11} \\ 2 G_0^{\frac{1}{2}} C' + 2bh + 2b'h' &= G_{12} \\ G_0^{\frac{1}{2}} D' + b^2 + b'^2 + b''^2 &= \tfrac{1}{2} G_{22} \end{aligned}\right\},$$

$$\left.\begin{aligned} G_0^{\frac{1}{2}} Q' + 3hB + 3h'B' &= \tfrac{1}{2} G_{111} \\ G_0^{\frac{1}{2}} R' + 2hC + bB + 2h'C' + b'B' + b''B'' &= \tfrac{1}{2} G_{112} \\ G_0^{\frac{1}{2}} S' + hd + 2bC + h'D' + 2b'C' + 2b''C'' &= \tfrac{1}{2} G_{122} \\ G_0^{\frac{1}{2}} T' + 3bD + 3b'D' + b''D'' &= \tfrac{1}{2} G_{222} \end{aligned}\right\};$$

and likewise from

$$x_1 x_2 + y_1 y_2 + z_1 z_2 = F(p, q) = 0,$$

we have

$$\left.\begin{aligned} \tfrac{1}{2} E_0^{\frac{1}{2}} B + ah + \tfrac{1}{2} G_0^{\frac{1}{2}} A' + a'h' &= 0 \\ E_0^{\frac{1}{2}} C + ab + h^2 + G_0^{\frac{1}{2}} B' + a'b' + h'^2 + a''b'' &= 0 \\ \tfrac{1}{2} E_0^{\frac{1}{2}} D + bh + \tfrac{1}{2} G_0^{\frac{1}{2}} C' + b'h' &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} E_0^{\frac{1}{2}} Q + 3aB + 3hA + G_0^{\frac{1}{2}} P' + 3a'B' + 3h'A' + 3a''B'' &= 0 \\ E_0^{\frac{1}{2}} R + 2aC + 3hB + bA + G_0^{\frac{1}{2}} Q' + 2a'C' + 3h'B + b'A' + 2a''C'' + b''A'' &= 0 \\ E_0^{\frac{1}{2}} S + aD + 3hC + 2bB + G_0^{\frac{1}{2}} R' + a'D' + 3h'C' + 2b'B' + a''D'' + 2b''B'' &= 0 \\ E_0^{\frac{1}{2}} T + 3hD + 3bC + G_0^{\frac{1}{2}} S' + 3h'D' + 3b'C' + 3b''C'' &= 0 \end{aligned}\right\},$$

These three sets, each of seven relations, will be called the *E*-set, the *G*-set, and the *F*-set respectively.

The first three equations of the *E*-set determine A, B, C. The first three equations of the *G*-set determine B', C', D'. The first equation of the *F*-set then determines A_1', and the third equation of the *F*-set determines D. The second equation of the *F*-set then becomes a relation among the quantities which have been determined. The respective values are

$$\left.\begin{aligned} E_0^{\frac{1}{2}} A &= \tfrac{1}{2} E_{11} - \frac{E_1^2}{4 E_0} - \frac{E_2^2}{4 G_0} - L_0^2 \\ E_0^{\frac{1}{2}} B &= \tfrac{1}{2} E_{12} - \frac{E_1 E_2}{4 E_0} + \frac{E_2 G_1}{4 G_0} \\ E_0^{\frac{1}{2}} C &= \tfrac{1}{2} E_{22} - \frac{E_2^2}{4 E_0} - \frac{G_1^2}{4 G_0} \\ E_0^{\frac{1}{2}} D &= -\tfrac{1}{2} G_{12} + \frac{E_2 G_1}{4 E_0} - \frac{G_1 G_2}{4 G_0} \end{aligned}\right\},$$

$$\left.\begin{aligned} G_0^{\frac{1}{2}} A' &= -\tfrac{1}{2}E_{12} - \frac{E_1 E_2}{4E_0} + \frac{E_2 G_1}{4G_0} \\ G_0^{\frac{1}{2}} B' &= \tfrac{1}{2}G_{11} - \frac{E_2^2}{4E_0} - \frac{G_1^2}{4G_0} \\ G_0^{\frac{1}{2}} C' &= \tfrac{1}{2}G_{12} + \frac{E_2 G_1}{4E_0} - \frac{G_1 G_2}{4G_0} \\ G_0^{\frac{1}{2}} D' &= \tfrac{1}{2}G_{22} - \frac{G_1^2}{4E_0} - \frac{G_2^2}{4G_0} - N_0^2 \end{aligned}\right\}.$$

When these values of C and B' are substituted in the second equation of the F-set, it becomes

$$L_0 N_0 + \tfrac{1}{2}(E_{22} + G_{11}) - \frac{E_2^2}{4E_0} - \frac{G_1^2}{4G_0} - \frac{E_1 G_1}{4E_0} - \frac{E_2 G_2}{4G_0} = 0,$$

which is Gauss's characteristic equation. The latter thus has been established merely from the expansions.

7. The remaining twelve equations, four out of each of the three sets, are not of themselves sufficient to determine the fourteen undetermined coefficients which they involve. But, as yet, no account has been taken of the possible expansions of L and N, or of the fact that M is zero, except only in determining the values of a'', h'', b''.

Writing V for $(EG)^{\frac{1}{2}}$, we have

$$VX,\ VY,\ VZ = y_1 z_2 - y_2 z_1,\ \ y_1 x_2 - z_2 x_1,\ \ x_1 y_2 - x_2 y_1,$$

where X, Y, Z are the direction cosines of the normal; and then

$$\begin{aligned} L &= X x_{11} + Y y_{11} + Z z_{11}, \\ 0 &= X x_{12} + Y y_{12} + Z z_{12}, \\ N &= X x_{22} + Y y_{22} + Z z_{22}. \end{aligned}$$

As Z is nearly equal to unity in the vicinity of the origin, and as we are finding z only up to the third order inclusive, so that z_{11}, z_{12}, z_{22} are taken only up to the first order, it follows that we shall require L, N, X, Y, Z only up to the first order. We at once have

$$\begin{aligned} VX &= -G_0^{\frac{1}{2}} a'' p = -G_0^{\frac{1}{2}} L_0 p, \\ VY &= -E_0^{\frac{1}{2}} b'' q = -E_0^{\frac{1}{2}} N_0 q, \\ VZ &= V_0 + (E_0^{\frac{1}{2}} h' + G_0^{\frac{1}{2}} a) p + (E_0^{\frac{1}{2}} b' + G_0^{\frac{1}{2}} h) q; \end{aligned}$$

and therefore

$$\begin{aligned} VL &= a\,(-G_0^{\frac{1}{2}}a''\,p) + a'\,(-E_0^{\frac{1}{2}}b''q) \\ &\quad + a''\,\{V_0 + (E_0^{\frac{1}{2}}h' + G_0^{\frac{1}{2}}a)\,p + (E_0^{\frac{1}{2}}b' + G_0^{\frac{1}{2}}h)\,q\} \\ &\quad + V_0\,(A''\,p + B''\,q), \\ 0 &= h\,(-G_0^{\frac{1}{2}}a''\,p) + h'\,(-E_0^{\frac{1}{2}}b''q) + V_0\,(B''\,p + C''\,q), \end{aligned}$$

and

$$\begin{aligned} VN &= b\,(-G_0^{\frac{1}{2}}a''\,p) + b'\,(-E_0^{\frac{1}{2}}b''q) \\ &\quad + b''\,\{V_0 + (E_0^{\frac{1}{2}}h' + G_0^{\frac{1}{2}}a)\,p + (E_0^{\frac{1}{2}}b' + G_0^{\frac{1}{2}}h)\,q\} \\ &\quad + V_0\,(C''\,p + D''\,q). \end{aligned}$$

Up to the first order of small quantities, we have

$$VL = V_0L_0\left(1 + \frac{E_1}{2E_0}\,p + \frac{E_2}{2E_0}\,q + \frac{G_1}{2G_0}\,p + \frac{G_2}{2G_0}\,q + \frac{L_1}{L_0}\,p + \frac{L_2}{L_0}\,q\right),$$

$$VN = V_0N_0\left(1 + \frac{E_1}{2E_0}\,p + \frac{E_2}{2E_0}\,q + \frac{G_1}{2G_0}\,p + \frac{G_2}{2G_0}\,q + \frac{N_1}{N_0}\,p + \frac{N_2}{N_0}\,q\right).$$

Substituting and equating coefficients, we find that the finite terms are equal, in virtue of the relations $a'' = L_0$, $b'' = N$. The remaining relations are

$$\left.\begin{aligned} -aa''G_0^{\frac{1}{2}} + a''(E_0^{\frac{1}{2}}h' + G_0^{\frac{1}{2}}a) + V_0A'' &= V_0L_0\left(\frac{E_1}{2E_0} + \frac{G_1}{2G_0} + \frac{L_1}{L_0}\right) \\ -a'b''E_0^{\frac{1}{2}} + a''(E_0^{\frac{1}{2}}b' + G_0^{\frac{1}{2}}h) + V_0B'' &= V_0L_0\left(\frac{E_2}{2E_0} + \frac{G_2}{2G_0} + \frac{L_2}{L_0}\right) \end{aligned}\right\},$$

which will be called the L-pair;

$$\left.\begin{aligned} -ha''G_0^{\frac{1}{2}} + V_0B'' &= 0 \\ -h'b''E_0^{\frac{1}{2}} + V_0C'' &= 0 \end{aligned}\right\},$$

which will be called the M-pair; and

$$\left.\begin{aligned} -ba''G_0^{\frac{1}{2}} + b''(E_0^{\frac{1}{2}}h' + G_0^{\frac{1}{2}}a) + V_0C'' &= V_0N_0\left(\frac{E_1}{2E_0} + \frac{G_1}{2G_0} + \frac{N_1}{N_0}\right) \\ -b'b''E_0^{\frac{1}{2}} + b''(E_0^{\frac{1}{2}}b' + G_0^{\frac{1}{2}}h) + V_0D'' &= V_0N_0\left(\frac{E_2}{2E_0} + \frac{G_2}{2G_0} + \frac{N_2}{N_0}\right) \end{aligned}\right\},$$

which will be called the N-pair. The L-pair determine values of A'' and B''; the N-pair determine values of C''

and D''; and then the M-pair become relations between the quantities involved. The respective values are

$$\left.\begin{aligned} A'' &= L_1 + \frac{L_0 E_1}{2E_0} \\ B'' &= L_2 - \frac{N_0 E_2}{2G_0} = \frac{L_0 E_2}{2E_0} \\ C'' &= N_1 - \frac{L_0 G_1}{2E_0} = \frac{N_0 G_1}{2G_0} \\ D'' &= N_2 + \frac{N_0 G_2}{2G_0} \end{aligned}\right\},$$

where the second values given for B'' and C'' are deduced from the M-pair.

Two equations therefore exist among the magnitudes of the surface. They are the two Mainardi-Codazzi relations, which thus are established merely from the expansions.

8. Now knowing the values of A'', B'', C'', D'', we return to the unused equations of the E-set, the F-set, and the G-set. They are twelve in number, and they contain ten undetermined coefficients, viz. P, Q, R, S, T; P', Q', R', S', T'. The E-set determine P, Q, R, S. The G-set determine Q', R', S', T'. The first of the F-set then determines P', and the fourth of that set then determines T. When the deduced values are substituted in the second and third of the F-set, these become relations among the magnitudes of the surface. The respective values are as follows:—

$$E_0^{\frac{1}{2}} P = \tfrac{1}{2} E_{111} - \tfrac{3}{4} \frac{E_1 E_{11}}{E_0} - \tfrac{3}{4} \frac{E_2 E_{12}}{G_0} + \tfrac{3}{8} \frac{E_1^3}{E_0^2} + \tfrac{3}{8} \frac{E_2^2 G_1}{G_0^2} - 3 L_0 L_1,$$

$$E_0^{\frac{1}{2}} Q = \tfrac{1}{2} E_{112} - \frac{1}{4E_0} (E_2 E_{11} + 2E_1 E_{12}) + \frac{1}{4G_0} (G_1 E_{12} + 2E_2 G_{11}) + \tfrac{3}{8} \frac{E_1^2 E_2}{E_0^2} - \tfrac{1}{8} \frac{E_2^3}{E_0 G_0} - \tfrac{1}{8} \frac{E_1 E_2 G_1}{E_0 G_0} - \tfrac{3}{8} \frac{E_2 G_1^2}{G_0^2} - \tfrac{1}{2} \frac{E_2}{E_0} L_0^2,$$

$$E_0^{\frac{1}{2}} R = \tfrac{1}{2} E_{122} - \frac{1}{4E_0} (2E_2 E_{12} + E_1 E_{22}) - \frac{1}{4G_0} (2G_1 G_{11} - E_2 G_{12}) + \tfrac{3}{8} \frac{E_1 E_2^2}{E_0^2} + \tfrac{1}{8} \frac{E_1 G_1^2}{E_0 G_0} + \tfrac{1}{8} \frac{E_2^2 G_1}{E_0 G_0} - \tfrac{1}{8} \frac{E_2 G_1 G_2}{G_0^2} + \tfrac{1}{4} \frac{G_1^3}{G_0^2} - \frac{G_1}{2G_0} L_0 N_0,$$

$$E_0^{\frac{1}{2}} S = \tfrac{1}{2} E_{222} - \tfrac{3}{4} \frac{E_2 E_{22}}{E_0} - \tfrac{3}{4} \frac{G_1 G_{12}}{G_0} + \tfrac{3}{8} \frac{E_2^3}{E_0^2} + \tfrac{3}{8} \frac{G_1^2 G_2}{G_0^2},$$

$$E_0^{\frac{1}{2}}T = -\tfrac{1}{2}G_{112} + \frac{1}{4E_0}(G_1E_{22} + 2E_2G_{12}) - \frac{1}{4G_0}(G_2G_{12} + 2G_1G_{22})$$
$$- \tfrac{3}{8}\frac{G_1E_2^2}{E_0^2} + \tfrac{1}{8}\frac{E_2G_1G_2}{E_0G_0} + \tfrac{1}{8}\frac{G_1^3}{E_0G_0} + \tfrac{3}{8}\frac{G_1G_2^2}{G_0^2} + \tfrac{1}{2}\frac{G_1}{G_0}N_0^2,$$

$$G_0^{\frac{1}{2}}P' = -\tfrac{1}{2}E_{112} - \frac{1}{4E_0}(2E_2E_{11} + E_1E_{12}) + \frac{1}{4G_0}(2G_1E_{12} + E_2G_{11})$$
$$+ \tfrac{3}{8}\frac{E_1^2E_2}{E_0^2} + \tfrac{1}{8}\frac{E_2^3}{E_0G_0} + \tfrac{3}{8}\frac{E_1^2E_2}{E_0^2} + \tfrac{1}{8}\frac{E_1E_2G_1}{E_0G_0} + \tfrac{1}{2}\frac{E_2}{E_0}L_0^2,$$

$$G_0^{\frac{1}{2}}Q' = \tfrac{1}{2}G_{111} - \tfrac{3}{4}\frac{E_2E_{12}}{E_0} - \tfrac{3}{4}\frac{G_1G_{11}}{G_0} + \tfrac{3}{8}\frac{E_1E_2^2}{E_0^2} + \tfrac{3}{8}\frac{G_1^3}{G_0^2},$$

$$G_0^{\frac{1}{2}}R' = \tfrac{1}{2}G_{122} - \frac{1}{4E_0}(2E_2E_{22} - G_1E_{12}) - \frac{1}{4G_0}(G_2G_{11} + 2G_1G_{12})$$
$$+ \tfrac{1}{4}\frac{E_2^3}{E_0^2} + \tfrac{1}{8}\frac{E_2G_1^2}{E_0G_0} - \tfrac{1}{8}\frac{E_1E_2G_1}{E_0^2} + \tfrac{3}{8}\frac{G_1^2G_2}{G_0^2} + \tfrac{1}{8}\frac{E_2^2G_2}{E_0G_0} - \tfrac{1}{2}\frac{E_2}{E_0}L_0N_0,$$

$$G_0^{\frac{1}{2}}S' = \tfrac{1}{2}G_{122} + \frac{1}{4E_0}(2G_1E_{22} + E_2G_{12}) - \frac{1}{4G_0}(2G_2G_{12} + G_1G_{22})$$
$$- \tfrac{3}{8}\frac{G_1E_2^2}{E_0^2} - \tfrac{1}{8}\frac{E_2G_1G_2}{E_0G_0} - \tfrac{1}{8}\frac{G_1^3}{E_0G_0} + \tfrac{3}{8}\frac{G_1G_2^2}{G_0^2} - \tfrac{1}{2}\frac{G_1}{G_0}N_0^2,$$

$$G_0^{\frac{1}{2}}T' = \tfrac{1}{2}G_{222} - \tfrac{3}{4}\frac{G_1G_{12}}{E_0} - \tfrac{3}{4}\frac{G_2G_{22}}{G_0} + \tfrac{3}{8}\frac{E_2G_1^2}{E_0^2} + \tfrac{3}{8}\frac{G_2^3}{G_0^2} - N_0N_2.$$

When these values are substituted in the second relation of the F-set, a condition among the magnitudes is established. Using the Mainardi-Codazzi expression for N_1, we can transform the condition to

$$\frac{\partial K}{\partial p} + \tfrac{1}{2}\frac{E_1}{E_0}K = 0,$$

where

$$K = 0$$

is the Gauss characteristic equation; that is, the condition is actually satisfied in virtue of the relations already established. When we proceed similarly with the third relation of the F-set, we can transform it to

$$\frac{\partial K}{\partial q} + \tfrac{1}{2}\frac{G_2}{G_0}K = 0;$$

that is, this condition also is actually satisfied in virtue of the relations already established.

9. Such are the expansions which I wish to put upon record. They can be used for various purposes; two examples follow.

10. The projection of the line of curvature OM, given by $q_0=0$, on the tangent plane at O is

$$y=\frac{a'}{2E_0}x^2+\frac{1}{6E_0^2}(E_0^{\frac{1}{2}}A-3aa')x^3$$
$$+\frac{1}{24E_0^3}(E_0P'-6E_0^{\frac{1}{2}}aA'-4E_0^{\frac{1}{2}}a'A+15a^2a')x^4+\ldots;$$

while the projection of the line of curvature ON, given by $p_0=0$, on the same plane is

$$x=\frac{b}{2G_0}y^2+\frac{1}{6G_0^2}(G_0^{\frac{1}{2}}D-3bb')y^3$$
$$+\frac{1}{24G_0^3}(G_0T-6G_0^{\frac{1}{2}}b'D-4G_0^{\frac{1}{2}}bD'+15bb'^2)y^4+\ldots.$$

11. Again, imagine a geodesic drawn from O to a point R and projected on the tangent plane at O, and let R' be the projection of R; the deviation of the tangent to the projection at O from the straight line OR' can be found as follows. Denoting by P and Q the current coordinates of a point on the geodesic, we know that the differential equation of the geodesic is

$$\frac{d^2Q}{dP^2}=-\frac{G_1}{2E}\left(\frac{dQ}{dP}\right)^3+\left(\frac{E_2}{2E}-\frac{G_2}{2G}\right)\left(\frac{dQ}{dP}\right)^2$$
$$+\left(\frac{E_1}{2E}-\frac{G_1}{G}\right)\frac{dQ}{dP}+\frac{E_2}{2G}.$$

The geodesic is to pass through O; and therefore, in the immediate vicinity of O, the most important terms in the integral of the equation are

$$Q=\mu P+HP^2,$$

where

$$2H=-\frac{G_1}{2E_0}\mu^3+\left(\frac{E_2}{2E_0}-\frac{G_2}{2G}\right)\mu^2+\left(\frac{E_1}{2E_0}-\frac{G_1}{G_0}\right)\mu+\frac{E_2}{2G_0}$$
$$=\alpha\mu^3+\beta\mu^2+\gamma\mu+\delta,$$

say, where α, β, γ, δ are the values of the respective coefficients at the place O. The geodesic is to pass through R; hence

$$q=\mu p+Hp^2;$$

so that we now have two equations to determine μ and H.

In the general case, p is not zero; so we have, as the most important terms,

$$2H = \alpha \frac{q^3}{p^3} + \beta \frac{q^2}{p^2} + \gamma \frac{q}{p} + \delta,$$

$$\mu = \frac{q}{p} - Hp;$$

thus the most important terms in the integral equation are

$$Q = \frac{q}{p} P + H(P^2 - Pp).$$

The equation of the tangent at the origin to the projection of the geodesic on the plane is

$$E_0^{\frac{1}{2}} p Y - G_0^{\frac{1}{2}} (q - Hp^2) X = 0.$$

When x and y are the coordinates of R', the equation of OR' is

$$xY - yX = 0.$$

Hence, if ϕ is the angle between them,

$$\tan\phi = \frac{G_0^{\frac{1}{2}} (q - Hp^2) x - E_0^{\frac{1}{2}} py}{E_0^{\frac{1}{2}} px + G_0^{\frac{1}{2}} (q - Hp^2) y}.$$

The most important terms in the numerator are

$$\tfrac{1}{2} \frac{b}{G_0} y^3 + \left(\frac{h}{E_0^{\frac{1}{2}} G_0^{\frac{1}{2}}} - \tfrac{1}{2} \frac{b'}{G_0} \right) xy^2 + \left(\tfrac{1}{2} \frac{a}{E_0} - \frac{h'}{E_0^{\frac{1}{2}} G_0^{\frac{1}{2}}} \right) x^2 y - \left(\tfrac{1}{2} \frac{a'}{E_0} + H \frac{G_0^{\frac{1}{2}}}{E_0} \right) x^3,$$

and the most important terms in the denominator are

$$x^2 + y^2.$$

If then we write

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have

$$\tan\phi = r \left(\tfrac{1}{2} \frac{b}{G_0}, \ \frac{h}{E_0^{\frac{1}{2}} G_0^{\frac{1}{2}}} - \tfrac{1}{2} \frac{b'}{G_0}, \ \tfrac{1}{2} \frac{a}{E_0} - \frac{h'}{E_0^{\frac{1}{2}} G_0^{\frac{1}{2}}}, \ \tfrac{1}{2} \frac{a'}{E_0} - H \frac{G_0^{\frac{1}{2}}}{E_0} \right) (\sin\theta, \cos\theta)^3,$$

Thus, firstly, ϕ is a small angle of the same order of magnitude as r; and, secondly, there are three directions along which ϕ is a small angle of the second order of magnitude for sufficiently small values of r.

INTERPOLATION AND QUADRATURE—SYMMETRICAL FORMS.

By *R. Hargreaves, M.A.*

§ 1. The material used here for interpolation consists of an equal number of entries above and below the position for which a value is sought. The position may be defined by a fraction x above the lower central term or y below the upper central, $x+y$ being 1; and if n pairs are used, the formula obtained is

$$\tfrac{1}{2}p_0-\alpha_1 p_2+\alpha_2 p_4 \ldots+(-1)^{n-1}\alpha_{n-1}p_{2n-2}$$
$$+(x-y)\left\{\tfrac{1}{2}D_1-\frac{\alpha_1}{3}D_3+\frac{\alpha_2}{5}D_5\ldots+(-1)^{n-1}\frac{\alpha_{n-1}}{2n-1}D_{2n-1}\right\}(1).$$

In the normal diagram of differences the original and derived columns possess a central pair or a single centre alternately; the latter are $D_1D_3D_5$ ..., the sums of the pairs are $p_0p_2p_4$ ..., the subscript in each case marking the order. The coefficients α depend on the product xy, viz.

$$\left.\begin{aligned}&\frac{\alpha_n}{\alpha_{n-1}}=\frac{(n-1)\,n+xy}{(2n-1)\,2n}\text{, true to }\frac{\alpha_1}{\alpha_0}=\frac{xy}{2}\text{, and consequently}\\&\alpha_1=\frac{xy}{4},\ \alpha_2=\frac{xy}{24}\left(1+\frac{xy}{2}\right),\ \alpha_3=\frac{xy}{120}\left(1+\frac{xy}{2}\right)\left(1+\frac{xy}{6}\right),\ldots\end{aligned}\right\}(2),$$

with a ratio of consecutive coefficients always less than $\frac{1}{4}$. If the entries are taken in the reverse order x and y are interchanged, and the signs of odd differences are changed, so that the second line of (1) is unaltered, and the whole formula symmetrical. The feature which differentiates the present formula from that of Stirling, is that the latter is based on a single centre, the former on a central pair, flanked in each case by an equal number of entries. This brings a substantial gain in practical utility, and accordingly coefficients are calculated in § 3 for application to a first decimal of the difference in tabular entries.

§ 2. Consider the extra terms introduced when we use n pairs $u_1 \ldots u_{2n}$ to find u_{n+x} instead of using $n-1$ pairs $u_2 \ldots u_{2n-1}$. In the first case we use the expansion of $(1+\Delta)^{n-1+x}u_1$ as far as $\Delta^{2n-1}u_1$, in the second that of $(1+\Delta)^{n-2+x}u_2$ as far as $\Delta^{2n-3}u_2$. This last is greater than the

terms of $(1+\Delta)^{n-2+x}(1+\Delta)\,u_1$ or $(1+\Delta)^{n-1+x}u_1$ carried as far as $\Delta^{2n-3}u_1$ by a term

$$\frac{(x+n-2)\ldots(x-n+2)}{(2n-3)\,!}\,\Delta^{2n-2}u_1,$$

which arises from the substitution of $(1+\Delta)\,u_1$ for u_2 in the term containing $\Delta^{2n-3}u_2$. Thus the extra terms we are seeking are

$$\frac{(x+n-1)\ldots(x-n+2)}{(2n-2)\,!}\,\Delta^{2n-2}u_1+\frac{(x+n-1)\ldots(x-n+1)}{(2n-1)\,!}\,\Delta^{2n-1}u_1$$

$$-\frac{(x+n-2)\ldots(x-n+2)}{(2n-3)\,!}\,\Delta^{2n-2}u_1$$

$$=\frac{(x+n-2)\ldots(x-n+1)}{(2n-2)\,!}\,\Delta^{2n-2}u_1+\frac{(x+n-1)\ldots(x-n+1)}{(2n-1)\,!}\,\Delta^{2n-1}u_1$$

$$=\frac{(x+n-2)\ldots(x-n+1)}{(2n-1)\,!}\{(2n-1)\,\Delta^{2n-2}u_1+(x+n-1)\,\Delta^{2n-1}u_1\}$$

$$=(-1)^{n-1}\,\frac{(n-2+x)\,(n-2+y)\ldots xy}{2\,(2n-1)\,!}$$

$$\times\{(2n-1)\,(2+\Delta)\,\Delta^{2n-2}u_1+(x-y)\,\Delta^{2n-1}u_1\}$$

if for x in the bracket $x+n-1$ we write $\frac{1}{2}+\frac{1}{2}(x-y)$. Thus we get

$$(-1)^{n-1}\,\frac{(n-2+x)\,(n-2+y)\ldots xy}{2\,(2n-1)\,!}$$

$$\times\{(2n-1)\,(\Delta^{2n-2}u_1+\Delta^{2n-2}u_2)+(x-y)\,\Delta^{2n-1}u_1\},$$

which is

$$(-1)^{n-1}\,\frac{(n-2+x)\,(n-2+y)\ldots xy}{2\,(2n-1)\,!}$$

$$\times\{(2n-1)\,p_{2n-2}+(x-y)\,D_{2n-1}\}\ldots\ldots(3).$$

Comparing this with (1)

$$\alpha_{n-1}=\frac{(n-2+x)\,(n-2+y)\ldots xy}{2\,(2n-2)\,!}$$

and

$$\frac{\alpha_n}{\alpha_{n-1}}=\frac{(n-1+x)\,(n-1+y)}{(2n-1)\,2n}=\frac{(n-1)\,n+xy}{(2n-1)\,2n}$$

as stated above.

§ 3. In making a single interpolation it is well to replace $\frac{1}{2}p_0 + \frac{1}{2}(x-y)D_1$ by the sum of $x \times$ upper central term and $y \times$ lower central term. When no great refinement is needed the approximate formula for u_{2+x} or u_{3-y} given $u_1 \ldots u_4$ is

$$xu_3 + yu_2 - \frac{xy}{4}p_2 \text{ or } xu_3 + yu_2 - \frac{xy}{4}\{u_1 + u_4 - (u_2 + u_3)\} \ldots (4);$$

which I have found very serviceable.

For use in constructing a table with one extra decimal the numerical work for $x = .9$ is the same as for $x = .1$; we should then keep the two types of term separate as in (1). For this purpose numerical values of the coefficients are given, and we use

$$\beta_1\beta_2 \ldots \text{ for } (x \sim y)^{\alpha_1}_{3},\ (x \sim y)^{\alpha_2}_{5},\ \ldots.$$

x

.1 or .9
$$\begin{cases} \alpha_1 = \frac{9}{400}, & \alpha_2 = \frac{3}{800}\left(1 + \frac{9}{200}\right), & \alpha_3 = \frac{3}{4000}\left(1 + \frac{9}{200}\right)\left(1 + \frac{3}{200}\right) \\ \beta_1 = \frac{3}{500}, & \beta_2 = \frac{3}{5000}\left(1 + \frac{9}{200}\right), & \beta_3 = \frac{3}{35000}\left(1 + \frac{9}{200}\right)\left(1 + \frac{3}{200}\right) \end{cases}$$

.2 or .8
$$\begin{cases} \alpha_1 = \frac{1}{25}, & \alpha_2 = \frac{1}{150}\left(1 + \frac{2}{25}\right), & \alpha_3 = \frac{1}{750}\left(1 + \frac{2}{25}\right)\left(1 + \frac{2}{75}\right) \\ \beta_1 = \frac{1}{125}, & \beta_2 = \frac{1}{1250}\left(1 + \frac{2}{25}\right), & \beta_3 = \frac{1}{8750}\left(1 + \frac{2}{25}\right)\left(1 + \frac{2}{75}\right), \end{cases}$$

.3 or .7
$$\begin{cases} \alpha_1 = \frac{21}{400}, & \alpha_2 = \frac{7}{800}\left(1 + \frac{21}{200}\right), & \alpha_3 = \frac{7}{4000}\left(1 + \frac{21}{200}\right)\left(1 + \frac{7}{200}\right) \\ \beta_1 = \frac{7}{1000}, & \beta_2 = \frac{7}{10000}\left(1 + \frac{21}{200}\right), & \beta_2 = \frac{1}{10000}\left(1 + \frac{21}{200}\right)\left(1 + \frac{7}{200}\right) \end{cases}$$

.4 or .6
$$\begin{cases} \alpha_1 = \frac{3}{50}, & \alpha_2 = \frac{1}{100}\left(1 + \frac{3}{25}\right), & \alpha_3 = \frac{1}{500}\left(1 + \frac{3}{25}\right)\left(1 + \frac{1}{25}\right), \\ \beta_1 = \frac{1}{250}, & \beta_2 = \frac{1}{2500}\left(1 + \frac{3}{25}\right), & \beta_3 = \frac{1}{17500}\left(1 + \frac{3}{25}\right)\left(1 + \frac{1}{25}\right), \end{cases}$$

.5
$$\alpha_1 = \frac{1}{16},\ \alpha_2 = \frac{3}{256},\ \alpha_3 = \frac{5}{2048},\ \alpha_4 = \frac{35}{65536}.$$

Quadrature. §4. I find that the normal quadrature problem may be transformed so that the result is expressed

in terms of D's or p's according as an odd or even number of ordinates is used. Thus the formulæ

$$\int_0^{2n} dx \left[1 + x\Delta + \frac{x(x-1)}{2!}\Delta^2 + \ldots + \frac{x\ldots(x-2n+1)}{2n!}\Delta^{2n} \right] u_1$$

and ...(5*a*)

$$\int_0^{2n-1} dx \left[1 + x\Delta + \frac{x(x-1)}{2!}\Delta^2 + \ldots + \frac{x\ldots(x-2n+2)}{(2n-1)!}\Delta^{2n-1} \right] u_1$$

...(6*a*)

are replaced by types

$$2n\,(u_m + a_1 D_2 + a_2 D_4 + \ldots + a_n D_{2n}) \ldots\ldots\ldots (5b)$$

and

$$\frac{2n-1}{2}\,(p_0 + b_1 p_2 + b_2 p_4 + \ldots + b_{n-1} p_{2n-2}) \ldots\ldots (6b)$$

respectively. The ordinates are supposed to be entered in tabular form and differences constructed. The first shows the middle ordinate u_m with central differences of even order; in the second p_0 is the sum of the two middle ordinates. A special feature is that the approximation proceeds by a double step; the number of coefficients is halved, and the calculation of the higher ones simplified.

As a lemma we require for (5*a*) the propositions

$$\int_0^{2n} x\ldots(x-2n)\,dx = 0, \quad \int_0^{2n} (x-1)\ldots(x-2n+1)\,dx = 0 \ldots. (7),$$

with a factor docked at each end. These follow from writing $y = 2n - x$, and transforming the integrals. The corresponding lemma required for (6*a*) is

$$\left.\begin{aligned} \int_0^{2n-1} (x-1)\ldots(x-2n+2)\,(2x-2n+1)\,dx = 0 \\ \int_0^{2n-1} (x-2)\ldots(x-2n+3)\,(2x-2n+1)\,dx = 0 \end{aligned}\right\} \ldots (8).$$

Here the transformation $y = 2n-1-x$ gives

$$\begin{aligned} &\int_0^{2n-1} x\,(x-1)\ldots(x-2n+2)\,dx \\ &= -\int_0^{2n-1} (y-2n+1)\ldots(y-1)\,dy \\ &= -\int_0^{2n-1} (x-1)\ldots(x-2n+1)\,dx, \end{aligned}$$

the equivalent of the first relation in (8).

§ 5. Now write

$$\frac{x\ldots(x-2n+1)}{2n!}\Delta^{2n}-\frac{(x-1)\ldots(x-2n)}{2n!}\Delta^{2n}$$
$$=\frac{(x-1)\ldots(x-2n+1)}{(2n-1)!}\Delta^{2n}$$

with similar formulæ and add, thus

$$\frac{x\ldots(x-2n+1)}{2n!}\Delta^{2n}+\frac{x\ldots(x-2n+2)}{(2n-1)!}\Delta^{2n-1}+\ldots$$
$$=\frac{(x-1)\ldots(x-2n)}{2n!}\Delta^{2n}+\frac{(x-1)\ldots(x-2n+1)}{(2n-1)!}E\Delta^{2n-1}$$
$$+\frac{(x-1)\ldots(x-2n+2)}{(2n-2)!}E\Delta^{2n-2}+\ldots$$

where $E=1+\Delta$. Leave the first two terms untouched, repeat the process for the rest and so on; then we get for the integrand of (5*a*)

$$\left\{\frac{(x-1)\ldots(x-2n)}{2n!}\Delta^{2n}+\frac{(x-1)\ldots(x-2n+1)}{(2n-1)!}E\Delta^{2n-1}\right\}$$
$$+\left\{\frac{(x-2)\ldots(x-2n+1)}{(2n-2)!}E\Delta^{2n-2}+\frac{(x-2)\ldots(x-2n+2)}{(2n-3)!}E^2\Delta^{2n-3}\right\}+\ldots$$

But in virtue of (7) the integrals vanish for the second of each pair of terms, and so (5*a*)

$$=\int_0^{2n}dx\left[\frac{(x-1)\ldots(x-2n)}{2n!}\Delta^{2n}+\frac{(x-2)\ldots(x-2n+1)}{(2n-2)!}E\Delta^{2n-2}+\ldots\right]u_1.$$

In the first integral write $x-2n=(x-n)-n$, quote (7), and thus

$$\int_0^{2n}(x-1)\ldots(x-2n)\,dx=\int_0^{2n}(x-1)\ldots(x-n)^2\ldots(x-2n+1)\,dx$$
$$=2\int_0^n z^2(z^2-1)\ldots\{z^2-(n-1)^2\}\,dz$$

on putting $z=x-n$. The value of (5*a*) is therefore

$$2\int_0^n dz\left[E^n+\frac{z^2}{2!}E^{n-1}\Delta^2+\frac{z^2(z^2-1)}{4!}E^{n-2}\Delta^4+\ldots\right.$$
$$\left.+\frac{z^2(z^2-1^2)\ldots\{z^2-(n-1)^2\}}{2n!}\Delta^{2n}\right]u_1$$

or

$$2\int_0^n dz\left[u_m+\frac{z^2}{2!}D_2+\frac{z^2(z^2-1)}{4!}D_4+\dots\right.$$
$$\left.+\frac{z^2(z^2-1^2)\dots\{z^2-(n-1)^2\}}{2n!}D_{2n}\right]\dots(9).$$

Proceeding in the same way for the integrand of (6*a*), the first step gives

$$\left\{\frac{(x-1)\dots(x-2n+1)}{(2n-1)!}\Delta^{2n-1}+\frac{(x-1)\dots(x-2n+2)}{(2n-2)!}E\Delta^{2n-2}\right\}$$
$$+\left\{\frac{(x-2)\dots(x-2n+2)}{(2n-3)!}E\Delta^{2n-3}\frac{(x-2)\dots(x-2n+3)}{(2n-4)!}E^2\Delta^{2n-4}\right\}+\dots$$

Here apply (8) writing

$$2x-2n+1=2(x-2n+1)+2n-1,$$

and therefore

$$\int_0^{2n-1}\frac{(x-1)\dots(x-2n+1)}{(2n-1)!}dx=-\int_0^{2n-1}\frac{(x-1)\dots(x-2n+2)}{2(2n-2)!}dx;$$

then use $E-\frac{1}{2}\Delta=\frac{1}{2}(E+1)$, with similar treatment for other pairs, leading to (6*a*)

$$=\frac{E+1}{2}\int_0^{2n-1}dx\left[\frac{(x-1)\dots(x-2n+2)}{(2n-2)!}\Delta^{2n-2}\right.$$
$$\left.+\frac{(x-2)\dots(x-2n+3)}{(2n-4)!}E\Delta^{2n-4}+\dots\right]u_1.$$

With $z=2x-2n+1$ we get the value of (6*b*)

$$=\frac{E+1}{2}\int_0^{2n-1}dz\left[\frac{(z^2-1)(z^2-3^2)\dots\{z^2-(2n-3)^2\}}{2^{2n-2}(2n-2)!}\Delta^{2n-2}+\dots\right.$$
$$\left.+\frac{z^2-1}{2^2(2!)}E^{n-2}\Delta^2+E^{n-1}\right]u_1$$

or

$$\int_0^{2n-1}\tfrac{1}{2}\left[p_0+\frac{z^2-1}{2^2(2!)}p_2+\dots\right.$$
$$\left.+\frac{(z^2-1)(z^2-3^2)\dots\{z^2-(2n-3)^2\}}{2^{2n-2}(2n-2)!}p_{2n-2}\right]dz\dots(10).$$

Forms (9) and (10) exhibit an essential symmetry which is not apparent in the original forms.

§ 6. The coefficients of (5*b*) are found on integration to be

$$a_1 = \frac{n^1}{3!}, \quad a_2 = \frac{n^2(3n^2-5)}{3(5!)}, \quad a_3 = \frac{n^3}{3(7!)}(3n^4-21n^2+28),$$

$$a_4 = \frac{n^3}{5(9!)}(5n^6-90n^4+441n^2-540), \ldots .$$

Numerical values for the opening terms are:—

3 ordinates, 2 spaces, $2\left(u_m+\frac{1}{6}D_2\right)$,

5 „ 4 „ $4\left(u_m+\frac{2}{3}D_2+\frac{7}{90}D_4\right)$,

7 „ 6 „ $6\left(u_m+\frac{3}{2}D_2+\frac{11}{20}D_4+\frac{41}{840}D_6\right)$,

9 „ 8 „ $8\left(u_m+\frac{8}{3}D_2+\frac{86}{45}D_4+\frac{92}{189}D_6+\frac{989}{28350}D_8\right)$.

For (6*b*) we find

$$b_1 = \frac{(2n-1)^2-3}{4!}, \quad b_2 = \frac{3(2n-1)^4-50(2n-1)^2+135}{8(6!)},$$

$$b_3 = \frac{3(2n-1)^6-147(2n-1)^4+1813(2n-1)^2-4725}{24(8!)},$$

with numerical values for the opening terms:—

4 ordinates, 3 spaces, $\frac{3}{2}\left(p_0+\frac{1}{4}p_2\right)$,

6 „ 5 „ $\frac{5}{2}\left\{p_0+\left(1-\frac{1}{12}\right)p_2+\frac{p_4}{8}\left(1+\frac{1}{18}\right)\right\}$,

8 „ 7 „ $\frac{7}{2}\left\{p_0+\frac{23}{12}p_2+\frac{611}{720}p_4+\frac{751}{8640}p_6\right\}$.

In the higher differences no doubt approximate values, fractional or decimal, might be used for the coefficients.

If in (5*a*) we added a term $x\ldots(x-2n)\,\Delta^{2n-1}/(2n+1)!$, retaining the integration limits 0 to $2n$, the new term would vanish, or the introduction of the next difference would not affect the value; therefore, *pro tanto*, the formula is more accurate. Boole points out this feature for the case of seven ordinates, but does not appear to have noticed that it is true for any odd number, and dependent on (7).

The integrals (7) and (8) may be generalised by taking an arbitrary difference d, and altering the initial factor. Then with $b-a=2nd$, formula (7) is replaced by

$$\int_a^b (x-a)\dots(x-b)\,dx=0,\ \int_a^b (x-a-d)\dots(x-b+d)\,dx=0,\dots(11);$$

while with $b-a=(2n-1)\,d$, (8) is replaced by

$$\left.\begin{aligned}&\int_a^b (x-a)\dots(x-b)\ (2x-a-b)\,dx=0\\&\int_a^b (x-a-d)\dots(x-b+d)\,(2x-a-b)\,dx=0\end{aligned}\right\}\dots(12).$$

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLII.

On Weierstrass's singular integral, and on a theorem of Lerch.

1. Theorem. *If*

(i) x, α, *and* $f(x)$ *are real*;
(ii) $f(x)$ *is summable in any finite interval of values of* x;
(iii) *the integral*

$$\int_{-\infty}^{\infty} e^{-Ax^2} f(x)\,dx$$

is convergent for some value of A (*and so for all greater values*):

then

$$(1.1)\qquad \lim_{\alpha\to 0}\frac{1}{\sqrt{(\pi\alpha)}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/\alpha} f(x)\,dx=f(\xi)$$

for almost all values of ξ, *and in particular wherever* $f(\xi)$ *is continuous*; *and the limit is equal to*

$$(1.2)\qquad \tfrac{1}{2}\{f(\xi+0)+f(\xi-0)\}$$

whenever this expression has a meaning.

The first accurate discussion of the formula (1.1) is that of Weierstrass*. Weierstrass supposes $f(x)$ uniformly continuous and bounded. The formula has been discussed since by Hobson† and Lebesgue‡. Hobson supposes $f(x)$ to be summable in any finite interval, and

$$\int_{-\infty}^{\infty} |f(x)|\,dx \tag{1.3}$$

to be convergent; but confines himself to values of ξ for which the expression (1.2) has a meaning. In a note added to his paper, and attributed to the suggestion of Bromwich, he replaces the condition that (1.3) should be convergent by the much less stringent condition that

$$|f(x)| < e^{A|x|}\,\S \tag{1.4}$$

for some value of A and all sufficiently large values of $|x|$.

The question whether (1.1) holds for almost all values of ξ was first discussed by Lebesgue. Lebesgue proved that this is so if $f(x)$ is summable in any finite interval, and (1.3) is convergent, or if $f(x)$ is summable in some finite interval including ξ, and bounded outside it. The last condition may naturally be replaced by the more general condition (1.4).

The work of Hobson and Lebesgue thus includes everything stated in my theorem except that I employ a very much more general condition as to the behaviour of $f(x)$ at infinity. The generalisation which I make is however a very material one. For it is plain that, unless my condition (iii) is satisfied, there can be no question of the existence of the limit in question. My conditions are therefore *necessary and sufficient*.

2. Hobson and Lebesgue both discuss Weierstrass's integral as a special case of a 'singular integral' of a very general

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* K. Weierstrass, 'Ueber die analytische Darstellbarkeit sogenannter willkurlichen Functionen reeller Argumente', *Berliner Sitzungsberichte* (1885), pp. 633–639, 789–805 (*Werke*, vol. iii., pp. 1–37). See also E. Borel, *Leçons sur les fonctions de variables réelles*, pp. 51 *et seq.*

† E. W. Hobson, 'On a general convergence theorem, and the theory of the representation of a function by series of normal functions', *Proc. London Math. Soc.*, ser. 2, vol. vi. (1908), pp. 349–395 (especially pp. 367–370).

‡ H. Lebesgue, 'Sur les intégrales singulières', *Annales de la Faculté des Sciences de Toulouse*, ser. 3, vol. i. (1909), pp. 25–118 (especially pp. 90–92).

§ The condition is stated in the form

$$|f(x)| < |x|^p e^{A|x|},$$

but the simpler form adopted here asserts neither more nor less.

With this condition, Hobson's result includes everything proved by Weierstrass. With the more stringent condition it did not, since Weierstrass's conditions do not imply the convergence of (1.3).

type. I shall therefore prove the theorem *ab initio* and without reference to their work. I begin by recalling a fundamental theorem of Lebesgue, that if $f(x)$ is summable then

$$(2.1) \qquad \phi(x) = \int_0^x |f(\xi+t)+f(\xi-t)-2f(\xi)|\,dt = o(x),$$

when $x \to 0$, for almost all values of ξ; and I shall prove that (1.1) is true for all such values of ξ.

It is obvious that, in proving this, we may suppose without loss of generality that $\xi = 0$ and $f(\xi) = 0$. Further we may write

$$f_1 = f \quad (x>0), \quad f_1 = 0 \quad (x<0),$$
$$f_2 = 0 \quad (x>0), \quad f_2 = f \quad (x<0),$$

so that

$$f = f_1 + f_2,$$

and discuss the contributions of f_1 and f_2 to the integral separately. And when we have made these formal simplifications we have to prove that *if*

$$(2.2) \qquad \phi(x) = \int_0^x |f(t)|\,dt = o(x)$$

when $x \to 0$, *then*

$$(2.3) \qquad J(\alpha) = \int_0^\infty e^{-x^2/\alpha} f(x)\,dx = o(\sqrt{\alpha})$$

when $\alpha \to 0$.

3. I observe first that *if*

$$\int_0^\infty e^{-Ax^2} f(x)\,dx$$

is convergent, and $x_0 \geq 0$, *then*

$$(3.1) \qquad \left| \int_{x_0}^\infty e^{-x^2/\alpha} f(x)\,dx \right| < K e^{-x_0^2/2\alpha},$$

where K *is independent of* x_0 *and* α, *for all sufficiently small values of* α. To prove this, choose α_0 so that $2A\alpha_0 < 1$. Then

$$\int_0^\infty e^{-x^2/2\alpha} f(x)\,dx$$

is uniformly convergent for $0 < \alpha \leq \alpha_0$. There is therefore a constant K such that

$$\left| \int_{x_0}^{x_1} e^{-x^2/2\alpha} f(x)\,dx \right| < K,$$

for $0 \leq x_0 \leq x_1$ and $0 < \alpha \leq \alpha_0$. But

$$\int_{x_0}^{\infty} e^{-x^2/\alpha} f(x)\, dx = \lim_{X \to \infty} \int_{x_0}^{X} e^{-x^2/\alpha} f(x)\, dx$$

$$= \lim_{X \to \infty} e^{-x_0^2/2\alpha} \int_{x_0}^{x_1} e^{-x^2/2\alpha} f(x)\, dx,$$

where $x_0 \leq x_1$. From this the truth of (3.1) follows immediately.

Now choose δ so that

$$\phi(x) < \eta x = \frac{\epsilon x}{\sqrt{\pi}} \tag{3.2}$$

for $0 < x \leq \delta$; and write

$$F(x) = \int_0^x f(t)\, dt,$$

so that

$$|F(x)| \leq \phi(x). \tag{3.3}$$

Dividing the range of integration in $J(\alpha)$ into the two parts $(0, \delta)$ and (δ, ∞), and integrating the first part of the integral by parts, we obtain

$$J(\alpha) = e^{-\delta^2/\alpha} F(\delta) + \frac{2}{\alpha} \int_0^{\delta} x e^{-x^2/\alpha} F(x)\, dx + \int_{\delta}^{\infty} e^{-x^2/\alpha} f(x)\, dx.$$

It follows from (3.1), (3.2), and (3.3) that

$$|J(\alpha)| < e^{-\delta^2/\alpha} \phi(\delta) + K e^{-\delta^2/2\alpha} + \frac{2\eta}{\alpha} \int_0^{\delta} x^2 e^{-x^2/\alpha}\, dx$$

$$< e^{-\delta^2/\alpha} \phi(\delta) + K e^{-\delta^2/2\alpha} + \tfrac{1}{2}\epsilon \sqrt{\alpha};$$

since

$$\frac{2\eta}{\alpha} \int_0^{\infty} x^2 e^{-x^2/\alpha}\, dx = 2\eta \sqrt{\alpha} \int_0^{\infty} u^2 e^{-u^2}\, du = \tfrac{1}{2}\epsilon \sqrt{\alpha}.$$

But, when δ has been fixed, we have obviously

$$e^{-\delta^2/\alpha} \phi(\delta) + K e^{-\delta^2/2\alpha} < \tfrac{1}{2}\epsilon \sqrt{\alpha}$$

for all sufficiently small values of α; and so

$$|J(\alpha)| < \epsilon \sqrt{\alpha}.$$

This proves the theorem, in so far as it asserts that (1) holds almost everywhere and in particular at all points of continuity.

In order to prove the last clause of the theorem we have only to replace (2.1) by

$$\phi(x) = \int_0^x |f(\xi + t) + f(\xi - t) - f(\xi + 0) - f(\xi - 0)|\, dt, \tag{3.4}$$

to suppose that $\phi(x) = o(x)$, and to make a few trivial changes in the preceding argument.

4. It was proved by Lerch* that *if $f(x)$ is continuous and*

$$\int_0^1 x^n f(x)\,dx = 0 \tag{4.1}$$

for all positive integral values of n, then $f(x)$ is identically zero. Lerch bases his proof on Weierstrass's theorem concerning the representation of a continuous function of a series of polynomials. Another proof of a more direct character had been indicated earlier by Stieltjes† and was made explicit and rigorous by Landau‡. A third proof was given independently by Phragmén.§

It has been proved, more generally, by Hobson,‖ that if $f(x)$ is any summable function, and (10) holds for all positive integral values of n, then $f(x) = 0$ except at a set of measure zero. Hobson, like Lerch, uses Weierstrass's theorem. The result may, as has been remarked by Lebesgue,¶ be deduced at once from the theorem of § 1 ; for, since the expansion

$$e^{-(x-\xi)^2/\alpha} = 1 - \frac{(x-\xi)^2}{\alpha} + \frac{(x-\xi)^4}{2!\,\alpha^2} - \dots$$

is uniformly convergent, we have

$$\int_0^1 e^{-(x-\xi)^2/\alpha} f(x)\,dx = 0$$

for all positive values of α. An alternative method would be to observe that, since the power series for $\cos 2n\pi x$ and $\sin 2n\pi x$ are uniformly convergent, all the Fourier constants of $f(x)$ must be zero, and then to use Lebesgue's extension of Fejér's theorem on Fourier series.

5. It should be observed that Lerch states his theorem in the form : *if*

$$J(a) = \int_0^\infty e^{-ax} f(x)\,dx$$

and

$$J(a) = 0$$

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* M. Lerch, 'Sur un point de la théorie des fonctions génératrices d'Abel', *Acta Mathematica*, vol. xxvii. (1903), pp. 339–351.

† *Correspondance d' Hermite et de Stieltjes*, vol. ii., pp. 337–339.

‡ E. Landau, 'Ueber die Approximation einer stetigen Funktion durch eine ganze rationale Funktion', *Rendiconti di Palermo*, vol. xxv. (1908), pp. 337–345. I am indebted to this paper for my references to Stieltjes and Phragmén.

§ E. Phragmén, 'Sur une extension d'un théorème classique de la théorie des fonctions', *Acta Mathematica*, vol. xxviii. (1904), pp. 351–368 (pp. 361–363).

‖ E. W. Hobson, 'The fundamental lemma of the calculus of variations', *Proc. London Math. Soc.*, ser. 2, vol. xi. (1911), pp. 17–28.

¶ *l.c.*, p. 104.

for

$$a = \alpha + n\beta \quad (\beta > 0\,;\ n = 0,\ 1,\ 2,\ \ldots),$$

then $J(a)$ *is identically zero and* $f(x)$ *is 'in general' zero.*

Lerch naturally supposes $f(x)$ integrable in Riemann's sense. If we suppose merely that $f(x)$ is summable and the integral convergent, then it is easy to show, by Lerch's argument, that $f(x) = 0$ except at a set of measure zero, and in particular at all points of continuity.

We may suppose without loss of generality that $\alpha > \beta$. Then $a > 0$, and, if we write

$$F(x) = \int_0^x f(t)\, dt$$

it is easy to show that

$$F(x) = o\,(e^{\alpha x})$$

and

$$J(a) = a \int_0^\infty e^{-ax} F(x)\, dx.^*$$

If now we put $e^{-\beta x} = y$ and

$$\psi(y) = y^{(\alpha-\beta)/\beta} F\left(\frac{1}{\beta} \log \frac{1}{y}\right),$$

then $\psi(y)$ is continuous for $0 \leq y \leq 1$ and

$$\int_0^1 y^n \psi(y)\, dy = 0$$

for all positive integral values of n. It follows that F is identically zero, and so that f is zero except at a set of measure zero.

Some very beautiful applications of Lerch's theorem have been made recently by Ramanujan.†

I intend in some succeeding notes to give some applications of the theorems discussed in this note to various interesting problems concerning definite integrals, in particular to the 'problème des moments' of Stieltjes and certain 'inversion formulæ' of Mellin and other writers. I have therefore thought it best to discuss the theorems in some detail, even where I have but little to add to them.

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* The proofs depend merely on partial integration, and I leave them to the reader.

† S. Ramanujan, 'Some definite integrals connected with Gauss's sums', *Messenger*, vol. xliv. (1915), pp. 75–81; 'On certain infinite series', *ibid.*, vol. xlv. (1916), pp. 11–15.

THEORY OF NUMBERS TABLES—ERRATA.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's College, London.

[The thanks of the author are due to Mr. H J. Woodall, A.R.C.Sc., for help in reading the proof-sheets.]

1. *Introduction.* The object of this Paper is to give a List of *Corrigenda* in a number* of important Mathematical Tables connected with the Theory of Numbers.

The Tables here dealt with are stated by short Titles along with the authors' names in the List below, arranged in order of the date of publication (or composition).

I. Euler, L.; *Pellian Equation*, 1732—1770.
II. Legendre, A. M.; (i) *Quadratic Forms*, 1783—1830;
(ii) *Pellian Equation*, ditto
III. Gauss, C. F.; *Number of Classes of Quadratic Forms*, 1801—07.
IV. Barlow, P.; *Pellian Equation*, 1811.
V. Degen, C. F.; *Pellian Equation*, 1817.
VI. Jacobi, C. G. F.; *Canon Arithmeticus*, 1839.
VII. Cayley, A.; (i) *Pellian Equation*, 1857.
(ii) *Quadratic Forms*, 1862.
VIII. Reuschle, C. G.; *Complex Primes*, 1875.
IX. Lucas, Ed.; *Quadratic Forms of* $(z^n-1)\div(z-1)$, &c., 1875—79.
X. Lloyd Tanner; *Complex Quintan Primes*, 1892—3.
XI. Roberts, C. A.; *Square Roots of Primes Development*, 1892.
XII. Bickmore, Chas.; *Pellian Equation*, 1893.
XIII. Cunningham, Allan J. C., Lt.-Col. R.E.;
(i) *Binary Canon*, 1900;
(ii) *Quadratic Partitions*, 1904.
XIV. Whitford, E. E.; *Pellian Equation*, 1912.

I. Euler, L. *Pellian Equation.* Tables of solution of

* Other papers by the present author giving Lists of Corrigenda in such Mathematical Tables have already appeared in this Journal, viz., *On Haupt-Exponent Tables*, in vol xxxiii. (1904), pp. 145—155; *Quadratic Partition Tables—Errata*, in vol. xxxiv. (1905), pp. 132—136. These Lists are not here repeated, except in cases where fresh Corrigenda have been found.

this Equation appear in several of Euler's Memoirs and Works.

i. *Commentationes Arithmeticæ*, vol. i., Petrop., 1849. A short Table is given in two Memoirs in this Work.

1°. page 8* [date 1732-3]—
Solutions of $q^2 - ap^2 = +1$, $[a \neq \square]$, up to $a = 68$. No Errata found.

2°. pages 335-6 [date 1759]—
Solutions of $p^2 - lq^2 = +1$, $[l \neq \square]$, up to $a = 99$.

Corrigenda.

page 336. $l = 33$, For $p = 33125$, Read $p = 66249$.
,, $l = 83$, For $p = 32$, Read $p = 82$.
,, $l = 85$, For $q = 30906$, Read $p = 30996$.

ii. *Elements of Algebra* [in German, date 1770]. English Translation by J. Hewlett, 3rd Ed., 1823.

page 360; Solutions of $m^2 - an^2 = +1$, $[a \neq \square]$, up to $a = 99$. No Errata found.

II. Legendre, A. M. There have been three Editions of the large Work containing the Tables here considered, viz.

1°. *Essai sur la Théorie des Nombres*, Paris, 1798.
2°. same Title 2nd Ed., Paris, 1808.
3°. *Théorie des Nombres*, 3rd Ed., 2 vols., Paris, 1830.

These Tables, viz. a number of short Tables of various 2^{ic} forms, and one of the Pellian Equation are placed at the end of the 1st and 2nd Editions, and at the end of Vol. I. of the 3rd Edition. The Tables are not paged.

The first seven Tables (I.–VII.) are of various 2^{io} forms: they are the *same in all* the Editions.

Tables I., II. give the simplest reduced forms of the ternary forms of positive determinants A, B as below—

Table I. of $Ly^2 + 2Myz + Nz^2$, wherein $A = M^2 - LN > 136$; 2 pages.
Table II. of $Ly^2 + Myz + Nz^2$, wherein $B = M^2 - 4LN > 305$; 1 page.

Tables III.–VII. give the linear and 2^{ic} forms of the divisors of the binary 2^{ic} forms as below:—

Table III.	of $t^2 - au^2$,	up to $a = 79$;	7	pages.
,, IV.	of $t^2 + au^2$,	,, $a = 105$, $a = 4n+1$;	4	,,
,, V.	of $t^2 + au^2$,	,, $a = 103$, $a = 4n+3$;	2	,,
,, VI.	of $t^2 + 2au^2$,	,, $2a = 106$, $a = 4n+1$;	2	,,
,, VII.	of $t^2 + 2au^2$,	,, $2a = 102$, $a = 4n+3$;	2	,,

* Of the three Tables here described, the only one mentioned in Cayley's *Report on Mathematical Tables* is this one (the smallest) see *Brit. Assocn. Report for* 1875, page 319.

The last Table in each Edition gives the *least* solution of *one* of the two Pellian Equations as below—

Ed.	*Tab.*	*Pellian Eqn.*	*Sol.*	*Condition*	*Limit*	*pp.*
1°	XII.*	$m^2 - an^2 = -1,\ a \neq \square$	$\frac{m}{n}$	when possible	$a < 1003$	12
,,	,,	$m^2 - an^2 = +1,$,,	,,	when $m^2 - an^2 \neq -1$	$a = 1003$	
2°	X.	$m^2 - An^2 = -1,\ A \neq \square$	,,	when possible	$a < 135$	1
,,	,,	$m^2 - An^2 = +1,$,,	,,	when $m^2 - An^2 \neq -1$	$a = 135$	
3°	X.*	$x^2 - Ny^2 = -1,\ N \neq \square$	$x : y$	when possible	$N < 1003$	8
,,	,,	$x^2 - Ny^2 = +1,$,,	,,	when $x^2 - Ny^2 \neq -1$	$N = 1003$	

In the first two Editions the solutions are printed in the form of a fraction $\left(\frac{m}{n}\right)$, whilst in the 3rd Edition they are printed in the form of a ratio $(x : y)$. In each of these Works the two equations are brought together into a single Table with Argument a (in Ed. 1, 2), or N (in Ed. 3), but no indication is given of the particular equation (*i.e.* with the − or + sign) to which the solution belongs: it is easy to identify the proper equation by merely examining the *units digits*, say (m_0, n_0, a_0) of (m, n, a), (m, n, A), (x, y, N) as the case may be; for it is obvious *at sight* whether

$$m_0^2 - a_0 n_0^2 = -1 \text{ or } +1.$$

Still the want of some distinguishing mark, and the omission of the more important equation with the +1 result are a most decided inconvenience.

In the 3rd Edition the solutions are arranged continuously up to $N = 198$; but, after this, the solutions in which m exceeds 10 figures up to 15 figures are placed in a separate page, and those in which m is still larger are placed on a special page at the end.

Corrigenda.

The seven Tables (I.–VII.) of Quadratic Forms are so closely alike in the three Editions that the Corrigenda† thereof can be conveniently presented together.

Ed.	*Tab.*	*Page.*	*Argt.*	*For*	*Read*
3	I.	3	$A = 99$	$y^2 - 90z^2$	$y^2 - 99z^2$
2. 3	II.	1	B	79	69
3	II.	1	B	174	173

* Neither of the two extensive Tables (in the 1st and 3rd Editions) are mentioned in Cayley's *Report on Mathematical Tables*: a few lines only are given to the short Table in the 2nd Edition; see *Brit. Assocn. Report for* 1875, p. 319.

† The Corrigenda in the 2nd Edition are mostly taken from J. P. Kulik's *Divisores Numerorum Series centena millia non excedentium*, &c., Gratii, 1825, p. xxvi.

Diviseurs Linéaires.

Ed.	*Tab.*	*Pge.*	*Formula.*	*Line.*	*For*	*Read*
3	III.	1	t^2-14u^2	1	$51x$	$56x$
2,3	,,	2	t^2-34u^2	2	...123...	...127...
2	,,	2	t^2-29u^2	1	1,3.5.9,...	1,5,7,9,...
1,2,3	,,	5	t^2-61u^2	1-8	7,11,23,31,35,43,51, 55,59,63,67,71,79,87, 91,99,111,115,139, 143,151,155,159, 175,191,207,211, 215,223,227	3,15,19,27,39,47,75, 83,95,103,107,119,123, 127,131,135,147,163, 167,171,179,182,195, 199,203,219,231,235, 239,243
1.2.3	,,	5	t^2-62u^2	2	...103...	...107...
1,2,3	,,	7	t^2-77u^2	2	...135,141,...	...135,137,141,...
,,	,,	7	,,	4	...255,...	Omit 255
,,	,,	7	,,	6	...53...	Omit 53
,,	,,	7	,,	8	...167,173,...	...167,171,173,...
1	IV.	3	t^2+77u^2	3	...41,73.89:113,117,...	...41,61,73:101,117,...
2	,,	3	,,	3	...41,73,89:113,113, 129,...	...41,61,73: 101,117 129,...
1,2	,,	3	,,	4	...145,149:173,241, 257,285,...	...145,153:173,237, 241,285,...
3	,,	3	,,	4	...173,241,...	...173,237.241,...
1,2,3	,,	3	,,	7	...103...	Omit 103
1	,,	3	,,	8	...115,159,...	...115,119,159,...
2,3	,,	3	,,	8	...119,199,...	...119,159,199,...
1,2,3	V.	2	t^2+91u^2	4	...7...	Omit 7
,,	,,	3	,,	5	...111,125,...	...111,115,125,...
2	VII.	2	t^2+78u^2	1	...79:102,121,...	...79:103,121,...

Corrigenda in the Pellian Solutions.

1st Edn. Table XII. is described in the heading as giving solutions of $m^2-an^2=\pm 1$: but, in the Table itself the Argument is everywhere printed N (instead of a).

There are Errata in the values of m or of n, or of both, for the 38 values of a (or N) shown below: this Abstract will suffice to *fix the position* of the Errata; it is not worth while to give the actual Corrigenda (some of which are very large numbers), as this Edition is now practically superseded by the 3rd Edn., in which most of the Errata have been corrected.

$N=$	133,	214,	236,	301,	307,	331,	343,	344,	355,	365,	397,	501,	526,
Err. in	m;	m,n;	m;	n;	m;	m;	m,n;	n;	m;	m;	m,n;	m;	mn;
$N=$	532,	613,	619,	629,	655,	664,	673,	694,	718,	722,	753,	771,	801,
Err. in	m;	m;	m,n;	m;	m,n;	m;	m;	m,n;	m;	m,n;	n;	m,n;	m,n;
$N=$	806,	809,	851,	856,	865,	871,	878,	886,	944,	965,	995,	1001,	
Err. in	m;	m,n;	m;	m,n;	m;	m,n;	m,n;	n;	m;	m,n;	m;	m;	

2nd Edn. Table X. In the Arguments (A). For $A=$ 22, 33, 24; Read 22, 23, 24. No Errata in the solutions (m, n).

3rd Edn. Table X.

Page	*N*	*For*	*Read*
1	94	$x=2543295$	2143295
1	116	$x=9301$	9801
1	149	$x=9303$	9305
3	308	$x=251$	351
4	479	$y=139591$	136591
4	629	$x=1850$	7850
5	667	$y=4147768$	4147668
7	271	$x=\ldots88600$	$\ldots83600$
7	749	$x=\ldots495$	$\ldots4895$
7	809	$x=483\ldots$	$433\cdots$
8	751	$x=\ldots25318960$	$\ldots24418960$
8	823	$x=2351704\ldots$	$235170474\cdots$

III. Gauss, C. F. *Tafel der Anzahl der Classen binarer quadratischen Formen*, in *C. F. Gauss, Werke*, Band II., 2nd Ed., Göttingen (1876), Nachlass, pp. 450–476.

The Table headed Nachlass—Determinantes Negativi (pages 450–475) gives the number (G) of genera and the number (say κ) of classes of all the *negative* determinants ($-D$) as follows, arranged in hundreds and thousands, viz.

All D in the 1st to 30th hundred, and in the 43rd, 51st, 61st, 62nd, 63rd, and also in the 91st to 100th, and 117th to 120th hundred.

All D in the 1st, 2nd, 3rd, and 10th thousand.

800 D of each of the forms $D=-(15n+7)$, $-(15n+13)$, >12000.

7 D of form $D=-15n$ between -999900 and -999975.

The determinants ($-D$) which possess the same number (G) of genera, and also the same number (κ) of classes per genus, are all placed together, in each hundred and in each thousand separately, with G, κ, as "Arguments"; thus

G VIII. 9 means 8 genera with 9 classes in each genus.

The two numbers in brackets ( ) *at head* of each G group indicate—

(1). The total number of determinants in that G group.

(2). The total number of classes of all those determinants.

The two numbers marked *Summa* at foot of each hundred, and of each thousand, indicate—

(1). The total number of genera in that hundred or thousand.

(2). The total number of (properly primitive) classes in that hundred or thousand.

[The total number of improperly primitive classes is shown *at the foot* of some of the groups of 100 or 1000 numbers, following the word *Impropr.*]

The "index of irregularity" of the few irregular negative determinants is shown by placing the index-number between

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* Most of these Errata will be found in *Journ. de Mathém. Elém.* (1886), p. 139; and (1887), p. 183, under *Correspondance de M. H. Richaud*; and in *Atti dell' Accad. Pont. d. Nuovi Lincei*, t. xx., Rome (1866), in a Note by M. E. Catalan.

asterisks inside brackets, thus (*3*). The total number of irregular determinants in each hundred or thousand group is shown at the foot of that group, preceded by the word *Irreg.*

The short Table (on pages 475, 476) headed DETERMINANTES POSITIVI gives the number (G) of genera, and number (say κ) of classes, of *positive* Determinants ($+D$) as follows, arranged in hundreds, viz.

All D in the 1st, 2nd, and 3rd hundreds (omitting squares).
All D in the 9th and 10th hundreds.
Four D between 300 and 800.

The arrangement and classification with "Arguments" G, κ is the same as described above for negative determinants.

The numbers in brackets ( ) at the head of each G group indicate the following totals :—

(1) The total number of determinants in that G group (in the 1st, 2nd, 9th, and 10th hundreds only).

(2) The total number of classes in that G group (given in the 10th hundred only).

These Tables were apparently composed in 1801–1807, but only published in 1876 in the Appendix to vol. ii. of Gauss's "Werke."

No explanation is given in the Text. A short description (of the principal elements only) will be found in the Editor's *Bemerkungen* on page 321; but there is none given of the abbreviations* used, nor of the mode of recording the totals.

A description of the Tables in some detail is given in Cayley's *Report on Mathematical Tables* on pages 528, 529 of the Report of the British Association for 1875.

Corrigenda.

On pages 522, 523 is a List of 36 Corrigenda in the Table of Negative Determinants by the Editor (Herr Schering).

In the *Bulletin of the American Mathematical Society*, vol. ix., 1902–3, page 467, is a List of 56 Corrigenda in the Table of Negative Determinants taken from a MS. by the late Prof. A. M. Nash. These include 10 of those already published in Schering's List (see above): and 33 relate to irregular determinants; these last are difficult to locate in Gauss's Table for want of sufficient references.

Here follows a List of all the Corrigenda† known to the

* For want of this it is difficult to understand the items—Propr. 3192, Propr. 3282 (at foot of page 453), Propr. 3416 (at foot of page 454).

† Those of the Negative Determinants are taken from Prof. Nash's List (with references added to locate the irregular determinants): those of the Positive Determinants are due to the present author.

present author, but excluding the 36 available in Schering's List on pages 522, 523 of the Work itself (see above).

Corrigenda in Table of Negative Determinants.

Page.	*Cent.*	*For*	*Read*
451	5	*G* II. 9 459*	*G* II. 9 459 (*3*)
,,	5	,, IV. 4 468	,, IV. 4 468 (*2*)
,,	5	,, IV. 4 485	,, IV. 5 485
,,	6	,, IV. 4 544	,, IV. 4 544 (*2*)
,,	6	,, I. 9 547	,, I. 9 547 (*3*)
,,	6	,, II. 9 557	,, II. 13 557
,,	7	,, I. 25 647	,, I. 23 647
452	9	,, IV. 6 894	,, IV. 7 894
,,	10	,, II. 9 931	,, II. 9 931 (*3*)
,,	10	,, IV. 3 933	,, IV. 4 933
,,	10	,, IV. 4 993	,, IV. 3 993
,,	12	,, IV. 9 1116	,, IV. 6 1116
453	13	,, II. 10 1261	,, IV. 5 1261
,,	14	,, I. 27 1367	,, I. 25 1367
,,	14	,, IV. 7 1396	,, II. 14 1396
454	16	,, IV. 8 1508	,, IV. 8 1508 (*2*)
,,	16	,, IV. 8 1598	,, IV. 8 1598 (*2*)
,,	17	,, II. 9 1683	,, II. 6 1683
,,	18	,, IV. 9 1701	,, IV. 9 1701 (*3*)
,,	18	,, VIII. 4 1725	,, VIII. 4 1725 (*2*)
,,	18	,, IV. 10 1796	,, II. 20 1796
455	19	,, IV. 9 1836	,, IV. 9 1836 (*3*)
,,	19	,, VIII. 4 1872	,, VIII. 4 1872 (*2*)
,,	20	,, IV. 8 1940	,, IV. 10 1940
456	22	Centas 2 (at top)	Centas 22
456	24	,, IV. 12 2331	,, IV. 12 2331 (*2*)
,,	22	,, IV. 12 2196	,, IV. 12 2196 (*2*)
,,	22	,, IV. 16 2180	,, IV. 16 2180 (*2*)
,,	24	,, IV. 8 2304	,, IV. 8 2304 (*2*)
,,	24	,, VIII. 4 2320	,, VIII. 4 2320 (*2*)
457	27	,, II. 33 2636	,, II. 33 2636 (*2*)
458	29	,, IV. 12 2900	,, IV. 12 2900 (*2*)
456	22	,, II. 9 2188	,, II. 9 2188 (*3*)
455	21	,, VIII. 5 2085	,, VIII. 4 2085
456	23	,, IV. 11 2204	,, IV. 13 2204
457	25	,, VIII. 4 2448 (*2*)	,, VIII. 4 2448
459	61	,, VIII. 8 6032	,, VIII. 8 6032 (*2*)
,,	61	,, IV. 24 6068	,, IV. 24 6068 (*2*)
,,	61	,, II. 27 6075 (*3*)	,, II. 27 6075 (*9*)
,,	61	,, IV. 12 6084	,, IV. 12 6084 (*2*)
460	62	,, IV. 8 6148	,, IV. 8 6148 (*2*)
,,	62	,, IV. 20 6176	,, IV. 20 6176 (*2*)
461	92	,, IV. 32 9104 (*2*)	,, IV. 32 9104
,,	92	,, VIII. 4 9108	,, VIII. 4 9108 (*2*)
,,	92	,, VIII. 8 9136	,, VIII. 8 9156 (*2*)
,,	94	,, VIII. 12 9324	,, VIII. 12 9324 (*2*)
462	96	,, VIII. 8 9513 (*2*)	,, VIII. 8 9513
,,	96	,, IV. 40 9554	,, IV. 40 9554 (*2*)

Corrigenda in Table of Positive Determinants.

Page.	*Cent.*	*For*	*Read*
475	1	*G* IV. 1 99	*G* IV. 2 99
,,	2	*G* IV. 1 136	*G* IV. 2 136
,,	2	*G* VIII. 1 150	*G* IV. 1 150
,,	2	*G* IV. 1 156	*G* IV. 2 156
,,	2	*G* II. 1 174	*G* IV. 1 174
,,	3	[at head of Table]	Excidunt 3
,,	3	omitted	*G* IV. 1 208
,,	3	omitted	*G* II. 1 209
,,	3	*G* II. 1 229	*G* II. 1 227
476	9	*G* IV. 1 850	*G* IV. 2 850
,,	9	*G* IV. 1 885	*G* IV. 2 885
,,	10	*G* IV. 1 904	*G* IV. 2 904

IV. Barlow, P. *Elementary Investigation of the Theory of Numbers*, London, 1811.

The Table on page 507 contains the Least Solution (p, q) of the Pellian Equation

$$p^2 - Nq^2 = +1, \ [N \neq \square], \text{ up to } N = 102.$$

No Errata found.

V. Degen, C. F. *Canon Pellianus*, &c., Hafniæ, 1817.

This Work contains Tables of the elements required in the solution of the Pellian Equations, and also the solutions of the same, as follows—

Table I. (pages 1—106) gives the elements $(r, d, \pm A)$ of the development of $\sqrt{a}$ in the form of a periodic continued fraction for *all* values of a (including $a = \square$) up to $a = 1000$. The elements are arranged in two lines: the upper line contains the integer portion (r) of $\sqrt{a}$ followed by the half-cycle of the denominators (d) of the continued fraction; the lower line contains certain "auxiliary* numbers" $(\pm A)$ obtained along with each d in this development, viz. those numbers for which the equation $y^2 - ax^2 = \pm A$ solvable. Below these two lines are printed the Least Solutions (x, y) of the Pellian Equation $y^2 - ax^2 = +1$ for the same range of a as above.

Table II. (pages 109—112) gives the Least Solutions (x, y) of the Pellian Equation $y^2 - ax^2 = -1$ (when possible) up to $a \not> 1000$ (but omitting the case of $a = \alpha^2 + 1$).

The Introduction (pp. i.—xxiv.) contains a description (in Latin) of the mode of computing the elements $(d, \pm A)$, and of computing the solutions (x, y) from them.

* This is the term used by Cayley, see *Brit. Assocn. Report for* 1875, page 73.

Corrigenda.

*In the cycles**—

page 88, $a=853$; the 10th entry in the upper line should be 14, not 15.
page 98, $a=929$; the cycle should be

30, 2, 11, 1, 2, 3, 2, 7, 5, (2, 2)
1, 29, 5, 40, 19, 10, 25, 8, 11, (23, 23)

In the solutions (x, y)—[*Argument a*].

Tab.	*page.*	*a.*	*Corrigenda.*
I.	17	238	For $x=1756$, Read †756.
,,	36	437	,, $y=4499$, ,, §4599.
,,	65	672	,, $y=327$, ,, §337.
,,	75	751	The last 7 figures of y should be ‡...4418960.
,,	84	823	Insert §47 after 235170 in value of y.
,,	96	919	The last 14 figures of y should be ‡3675978149958 9.
,,	100	915	For $y=27551$. Read §275561.
,,	100	951	For $y=22420806$, Read §224208076.

Table II., page 107.

The Title of this Table states that the solution of $y^2-ax^2=-1$ is given when possible; but, as a matter of fact, the case of $a=a^2+1$ is everywhere omitted.

VI. Jacobi, C. G. J. *Canon Arithmeticus*, &c., Berlin, 1839.

This Work opens with a long "Introduction" (38 quarto pages containing a discussion, pages ii.–xviii.) of the number of primes <2500 of which the bases ± 10 are either a primitive root, or a residue of order 2, 3, 4, &c. This depends on an actual count of such primes taken from Burckhardt's Table‖ entitled *Grandeur de la période de la fraction décimale égale à* $1/p$, which gives the length of that period (= the Haupt-Exponent of 10) for all primes $p<2550$, and for 22 others <10000.

This Table (as printed in this *Canon*, pages iii.–v.) contains the 10 Errata¶ shown below along with the corrections needed.

Valeur de p.....		911	1213	1597	1831	1951	1993	2311	2437	3467
Grandeur de la période	For	450	1212	266	915	390	1992	462	2436	3466
	Read	455	202	133	305	195	664	231	1218	1733

The prime misprinted 3462 should also be corrected to 3467.

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* See Mr. C. A. Roberts's Table in the *Mathematical Magazine*, Washington, 1893, p. 106.

† See page 112 of Degen's Table.

‡ See *Journ. de Math. Elémen.*, 1887, p. 183; *Correspondance de M. Richaud.*

§ These 5 Corrigenda are due to the Author.

‖ See *Table des diviseurs pour tous les nombres du premier million*, by J.-Ch. Burckhardt, Paris, 1817, page 114.

¶ Previously published in the Author's paper *On Haupt-Exponent Tables*, in *Messenger of Mathematics*, vol. xxxiii, 1904, p. 150.

These Corrigenda affect* the small Tables on pages vi., vii., viii., x., xi., xii., xiii., xiv., xvi., xvii., which are compilations from Burckhardt's Table, and also affect slightly the *counts** of the number of primes throughout the Text (pages vi.—xxii.); the argument in the Text is however scarcely affected at all.

The Canon itself fills 240 quarto pages. It consists of two Tables of equal length, placed side by side, for every prime p and prime-power p^κ up to p and $p^\kappa \Rightarrow 1000$. In each Table some primitive root (g) of the prime p is selected as the "Base" of the Table.

The left-hand Table [Tabula Numerorum] gives, to Argument $I.$, the complete set of Least + Residues (N) of every power (g^I) of the Base (g) after division by the modulus (p or p^κ), from $I=1$ to $I=p-1$, in the case of a prime modulus (p), or up to $p^{\kappa-1}(p-1)$ in the case of a prime-power modulus (p^κ). [Here I means Index.]

The right-hand Table [Tabula Indicum] is the converse of the former; it gives, to Argument $N.$, the complete set of Indices (I) of the powers (g^I) whose Least + Residues after division by the prime (p) or prime-power (p^κ) are the numbers N.

The primitive root (g) used as the "Base" in each Table is usually ± 10 ($+10$ in 59 cases, -10 in 28 cases out of a total of 167 primes <1000); when these fail, some other primitive root is used, apparently the one most easily found, but in many cases the root (g) used is inconveniently large (*e.g.* $g=937$, 981) for arithmetical calculations.

The Canon is followed by a Table of 222 Corrigenda (covering 5 quarto pages, 243—248); of these 66 are in the left-hand Table and 156 in the right-hand Table. The reason of the great excess in the latter Table is that the left-hand Table is a direct computation of *successive* Residues, ending always with $g^{p-1} \equiv +1$, or $g^{p^{\kappa-1}(p-1)} \equiv +1$, a real Test of the work; whereas the right-hand Table is merely a re-arrangement of the former in a quite different order, containing no test of its correctness.

The following Table is a List of *additional*† Corrigenda which have since been discovered.

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* It does not seem worth while detailing these effects here; the reader can easily do this for himself. To give them in detail would take a good deal of space.

† Several of these were supplied to the present Author by the late Prof. Lloyd Tanner.

Corrigenda.

Tabula Numerorum.

Pag.	*p*	*Arg.*	*Loco*	*Lege*
62	457	453	325	320
225	243	12	206	208
228	361	131	169	165
234	841	192	233	223
193	929	Col. I.	91	90
193	929	,,	92	91
193	929	,,	93	92

Valor $(p-1)$.

Pag.	*p*	*Arg.*	*Loco*	*Lege*
63	461	$p-1$	2^4	2^2
76	523	,,	3.87	$3^2.29$
77	523	,,	3.87	$3^2.29$
219	997	,,	2^3	2^2

Tabula Indicum.

Pag.	*p*	*Arg.*	*Loco*	*Lege*
139	757	565	168	468
222	25	14	8	6
224	169	33	41	71
228	361	122	43	93
228	361	216	87	78
228	361	353	144	174
232	729	196	204	304
234	841	353	394	694
237	961	Col. N	61	61
			61	62

Cancel this Correction :

245	571	109	190	109

VII. Cayley, A. *Pellian Equations.*

Two numerical Tables are here reported on, originally published in Crelle's *Journal für die reine und angewandte Mathematik*, t. liii., 1857, pages 369–371, and t. lx., 1862, pages 157–172. Republished in Cayley's *Collected Mathematical Papers*, Cambridge, vol. iv., 1891, page 42, and vol. v., 1892, pages 144–156. Here quoted from the *Collected Papers*.

i. *Solutions de l'équation* $x^2 - Dy^2 = \pm 4$, in vol. iv.

Corrigenda.

page 42, $D=253$; For $x, y = 1177, 74$; Read $x, y = 1861, 117$.
,, $D=597$; For $x = 7949$; Read $x = 9749$.
,, $D=917$; For $y = 31$; Read $y = 39$.

ii. *Tables des formes quadratiques binaires*, &c., in vol. v. These Tables give, for every determinant $D < 100$ (both $-$ and $+$), the elements (a, b, c) of all the classes, and the quadratic characters $(\alpha, \beta, \delta, \epsilon, \delta\epsilon)$ thereof, and also the complete period of each class of the $+D$.

Corrigenda.

Page 144, $D=-17$; Erase the long bar under 1, 0, 17.
,, $D=-20$; For 2, 0, 5; Read 4, 0, 5.
,, $D=-34$; For 7, -1, 7; Read 5, -1, 7.

Page 145, $D=-40$; Insert a short bar under 0, 40.
,, ,, Insert a short bar under 0, 8.
,, ,, In Cols. of δ, ε; Enter + + in line 1, − − in line 2,
,, + − in line 3, − + in line 4.
,, ,, Cancel all entries in Col. of $\delta\varepsilon$.
,, $D=-56$; For 2, −1, 19; Read 3, −1, 19.
Page 147, $D=-88$; Insert a short bar under 0, 88.
,, ,, Insert a short bar under 0, 11.
,, ,, In Col. of δ; Enter +, −, −, + in lines 1, 2, 3, 4.
Page 149, $D=29$; Line 2; The period should be 2, 5, −2, 5, 2.
,, $D=37$; Line 3; Reverse the period, thus −3, 5, 4, 3, −7, &c.
,, $D=41$; Line 2; The period should be
2,5,−8,3,4,5,−4,3,8,5,−2,5,8,3,−4,5,4,3,−8,5,2.
Page 150, $D=50$; In Col. of ε; Enter + in line 1, + in line 2.
,, ,, Cancel the entries in Col. of $\delta\varepsilon$.
Page 151, $D=65$; Line 1; The period should be 1,8,−1,8,1.
Page 152, $D=91$; Line 2; For 3,7,−14; Read 3,7,−14.

VIII. Reuschle, C. G. *Tafeln complexer Primzahlen*, &c., Berlin, 1875.

It is almost impossible to give any sufficient* explanation of the Contents of this great Work (which runs to 671 quarto pages) in any short compass.

It contains numerous Congruence-Tables giving the roots (α, η, ϖ, w) of Congruences of various types arising from the complex roots of unity of order λ (a prime) or n a (composite), and of lower orders arising from the factors of $\lambda-1$ and n.

The simplest of these is

$$\text{M.A.P.F. of } (a^{\lambda}-1)\equiv 0 \pmod{p=m\lambda+1},\ \text{a prime},$$

where M.A.P.F. means *Maximum irreducible algebraic prime factor*.

Others of orders μ (a factor of $\lambda-1$ or n) are—

$\eta^2+c\equiv 0$, $\eta^4+c^2\equiv 0$, $\eta^8+c^4\equiv 0 \pmod{p}$; [$c$ various, but $\not> +13, -23$],
$\eta^2\pm\eta+c\equiv 0 \pmod{p}$, [$c=-1$ to -26 and $+2$ to $+21$],
$\eta^3\pm\eta^2+c\eta+d\equiv 0$, $\eta^4\pm\eta^3+c\eta^2+d\eta+c\equiv 0 \pmod{p}$, [$c, d \not> 32, 79$],

and many others too numerous to detail.

These Tables can be used for finding factors (p) of numbers (N) of the forms specified by the Congruences, but this use is limited by the small range of the divisors ($p \not> 1000$), and also by the omission of certain necessary† Tables.

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* This is admitted by the late Prof. A. Cayley. In his *Report on Mathematical Tables* he gives a description in some detail of Part I. only of this Work; see *Brit. Assocn. Report for* 1875, pages 334–336.

† *E.g.* On pages 444, 445 are given Congruence-Tables of $\varpi^2\mp 2\equiv 0 \pmod{p}$ for primes of form $p=8m+7$, 3 respectively, but not for primes $p=8m+1$.

Most of these Congruence-Tables are followed by Tables giving the values of the complex factors of p in their *simplest* form in the fields of λ or n, and in the sub-fields (μ) belonging to λ, n, and sometimes also in their *primary* form, and (in a few cases) those of p^2 and p^3; also in many cases, when the complex factors are ideal, the power of that factor which becomes actual is also given. These Tables are perhaps the most valuable part of this Work in the sense of the Theory of Numbers.

The range of the exponents (λ, n) of the fields is

λ an odd prime, or prime-power <100, n an odd composite $\not> 105$,
$n = 2^k \not> 128$, $n = 4m$ an even composite $\not> 120$.

All the Tables are very complete in the lower values of λ, n, and very incomplete in the higher values of the same.

Corrigenda.

Page.	λ, n.	*Tab*	p.	*For*	*Read*	*Auth.*†
2	5	I.	691	$\alpha = +220$	$+320$	
8	11	I.	199	$\alpha^3 = -69$	-60	
8	11	,,	199	$\alpha^{10} = -73$	-78	
8	11	,,	331	$\alpha^4 = +55$	$+85$	
8	11	,,	661	$\alpha^3 = -214$	-204	
193	15	I.	881	$p = 881$	811	
199	21	I.	463	$\omega^{10} = -44$	-14	
226	39	I.	541	$p = 541$	547 (in 3 places)	
239	45	I.	631	$\omega^{43} = +71$	$+121$	Cr
239	45	,,	631	$\omega^{44} = +223$	-11	Cr
273	57	I.	457	$\omega^{46} = -7$	-6	
273	57	,,	457	$\omega^{52} = +230^*$	-227	
285	63	I.	379	$\omega^{13} = -132$	-112	Cr
285	63	,,	757	$\omega^{13} = +202$	-202	Cr
285	63	,,	631	$\omega^{20} = -26$	-24	Cr
285	63	,,	757	$\omega^{23} = -203$	-183	Cr
285	63	,,	883	$\omega^{24} = +18$	-355	Cr
446	16	I.	113	$\omega = -43$	-48	
450	32	I.	257	$\omega = +85$	$+15$	
461	128	I.	641	$\omega^{19} = -275$	-305	
461	128	,,	769	$\omega^{29} = -138$	-38	
476	24	I.	601	$\omega^{11} = -306^*$	$+295$	
495	40	I.	641	$\omega^{19} = +324^*$	-317	
513	48	I.	97	$\omega^{17} = -10$	-11	
513	48	,,	337	$\omega^5 = -174^*$	$+163$	
513	48	,,	337	$\omega^{11} = +57$	$+38$	
513	48	,,	337	$\omega^9 = -154$	-153	
533	56	I.	673	$\omega^{27} = -83$	-85	Cr
635	96	I.	577	$\omega^{23} = -197$	-196	
643	100	I.	701	$\omega^{19} = -353^*$	$+348$	
643	100	,,	601	$\omega^{37} = -46$	-26	
643	100	,,	601	$\omega^{19} = +341$	$+241$	

* The numbers marked with a star (*) are *not really incorrect* in Reuschle's Table; but, being $> \frac{1}{2}p$, they violate Reuschle's rule of recording only roots $< \frac{1}{2}p$; corrected roots $(< \frac{1}{2}p)$ are therefore given.

† See Note [illegible] at foot of this Table on next page.

Page.	λ, *n.*	*Tab.*	*p.*	*For.*	*Read.*	*Auth.*
3	5	I.	601	$5-a^2+2a^4$	$5-a^3+2a^4$	T
3	5	,,	751	$12+2a+5a^2+9a^3$	$12+2a+8a^2+9a^3$	T
3	5	,,	821	$4+4a-4a^2+3a^3$	$1+4a-4a^2+3a^3$	T
3	5	,,	881	$4-5a+5a^4$	$4-5a^2+5a^4$	T
5	7	I.	491	a^3+3a^6	Cancel this entry	B, W
5	7	,,	547	$2-a+2a^5+2a^6$	a^3+3a^6	B, W
37	29	V.	.	$6+\eta_1$	$8+\eta_1$	W
106	43	.	.	λ = 43 [at top]	λ = 67	
108	67	VI.	.	Tab. VI.	Tab. VIII.	
108	67	VI.	.	$p=2, 7, 11, 31$	$p=2, 7, 11, 13, 31$	
176	25	I.	401	$f(a)=1-a^3-a^4$	$fa=1-a^3+a^4$	W
187	49	.	.	λ = 89 [at top]	λ = 49	
249	51	.	.	[line 4] 107.409	103.409	W
282	57	VI.1	.	$\varpi^2+\varpi-14=0$	$\varpi^2-\varpi-14=0$	
487	28	III.	.	$\varpi^4-2\varpi^2+4=0$	$\varpi^4-3\varpi^2+4=0$	
511	44	VI.	.	Tab. VI. [line 2]	Tab. IV.	
511	44	IV 1	.	$p=40m-5, +7$	$p=44m-5, +7$	
546	56	IV.4	.	$\varpi^4-49=0$	$\varpi^4+49=0$	
621	68	.	.	$n=68$ [at top]	$n=88$	
628	76	.	.	$n=76$ [at top]	$n=88$	

☞ The initials in the column headed "Auth." indicate the authority for the Corrigendum. B for Chas. E. Bickmore; Cr for T. G. Creak; T for H. W. Lloyd Tanner; W for A. E Western. The rest are due to the present Author.

IX. Lucas, Ed. *Quadratic Forms of* $(z^n-1)\div(z-1)$, &c.

There have been three issues of these Tables in the following Memoirs:—

i. *Sur les formules de Cauchy et de Legendre-Dirichlet* in the *Comptes Rendus de l'Assocn. Française pour l'avancement des Sciences*, Paris, 1878, pages 168–172.

ii. [Same Title, and same date as above], pages 1–10; a separate reprint of No. 1.

iii. *Sur la Série récurrente de Fermat*, Rome, 1879, page 6.

Memoirs i. *and* ii.

The notation is

Let X_n = Max. algebraic prime (irreducible) factor of (z^n-1). Then X_n, X_{4n} may be expressed in the following ways, viz.

$$4X_n = Y^2 - (-1)^{\frac{1}{2}(n-1)} \cdot nz^2Z^2 \quad \ldots\ldots(1),$$

$$X_n = Y_1^2 - (-1)^{\frac{1}{2}(n-1)} \cdot nzZ_1^2 \quad \ldots\ldots(2),$$

$$X_{4n} = Y_2^2 - (-1)^{\frac{1}{2}(n-1)} \cdot 2nzZ_2^2 \quad \ldots\ldots(3),$$

where Y, Y_1, Y_2, Z, Z_1, Z_2 are polynomials in z of form

$$Y = b_0z^\mu + b_1z^{\mu-1} + b_2z^{\mu-2} + \ldots + b_{\mu-1}z + b_\mu,$$

$$Z = \quad c_1z^{\mu-1} + c_2z^{\mu-2} + \ldots + c_{\mu-1}z,$$

wherein $\mu = \frac{1}{2}$ Totient of n, and Y_1, Y_2, Z_1, Z_2 are polynomials similar to Y, Z (but with different coefficients b_r, c_r).

There are three Tables in each of the Memoirs giving the values of

$$\mu;\ b_0, b_1, b_2, \ldots, b_{\frac{1}{2}\mu-1} \text{ or } b_{\frac{1}{2}\mu};\ c_1, c_2, \ldots, c_{\frac{1}{2}\mu-1}, \text{ or } c_{\frac{1}{2}\mu}$$

in the formulæ for Y, Z; Y_1, Z_1; Y_2, Z_2 respectively; only one half of the full number of b_r, c_r are tabulated, because in each case

$$b_r = \pm b_{\mu-r}, \text{ and } c_r = \pm c_{\mu-r},$$

and simple rules* are given for the $\pm$ signs.

The Table on page 165 of Memoir i., and on pages 2, 3 of Memoir ii., gives the values of μ, b, c in the Y, Z of formula (1) for the following values of n,

$$n = 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 61;$$

The Table on page 168 of Memoir i., and on pages 5, 6 of Memoir ii., gives the values of the μ, b, c in the Y_1, Z_1 of formula (2) for the following values of n,

$$n = 3, 5, 7, 11, 13, 15, 17, 19, 21, 23, 29, 31, 33, 35, 37, 39, 41, 57, 69, 105.$$

The Table on pages 170, 171 of Memoir i., and on pages 8, 9 of Memoir ii., gives the values of the μ, b, c in the Y_2, Z_2 of formula (3) for the following values of $2n$,

$$2n = 6, 10, 14, 22, 26, 30, 34, 38, 42, 46, 58, 62, 66, 70, 78, 102, 114.$$

Memoir iii. The Table on page 6 is of a somewhat more general kind than the above. It contains the values of Y, Z in the formula

$$X_n = Y_n^2 - n\,xy\,Z_n^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where X_n = Max. algebraic prime (irreducible) factor

of $x^n - y^n$, when $n = 4i+1$,

of $x^n + y^n$, when $n = 4i+3$, or $4i+2$,

for the following values of n,

$$n = 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34.$$

These formulæ are written out at full length when $n = 3$ to 14 and 21; but only one half of the formula is given when $n = 17$, or 19, and from 23 to 34, the other being left to be inferred from the symmetry ($b_r = b_{\mu-r}$, $c_r = c_{\mu-r}$), as these formulæ are quite symmetric.

The computing of these formulæ, especially of Nos. 2, 3, 4, is so laborious that it is convenient to have a simple Test for the coefficients b_r, c_r thereof.

Take $z = +1$ in formulæ (1), (2), (3), and $y = +1$ in

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* But see the Corrigendum on these rules below (page 65).

formula (4). Let p denote an *odd* prime, and let (τ', υ'), (τ, υ), (f', g'), (f, g) be solutions of the Pellian Equations as follows:—

$$\tau'^2 - n\upsilon'^2 = -1, \quad \tau^2 - n\upsilon^2 = +1,$$
$$f'^2 - ng'^2 = -4, \quad f^2 - ng^2 = +4.$$

Then X_n, X_{4n} reduce to either n or 1 according to the form of n; Y, Z reduce to $\Sigma(b)$, $\Sigma(c)$ in each of the formulæ, and become solutions of the above Pellian Equations, as shown in the scheme below [but write $2n$ for n therein when testing formula (3)]. They are usually the minimum solutions [except when n in formulæ (1), (2), (4), or $2n$ informula (3) is non-idoneal*]. This provides a simple Test for the sums $\Sigma(b)$, $\Sigma(c)$ of the coefficients.

n	*Form* (1) X_n Y Z	*Form* (2) X_n Y_1 Z_1	*Form* (3) X_{4n} Y_2 Z_2	*Form* (4) X Y Z
$4i+3=p$	n, 0, ± 2	n, 0, ± 1	1, $\pm\tau$, $\pm\upsilon$	1, $\pm\tau$, $\pm\upsilon$
$4i+3=p$	1, ± 2, 0	1, ± 1, 0		1, $\pm\tau$, $\pm\upsilon$
$4i+1=p$	n, $\pm ng'$, $\pm f'$	n, $\pm n\upsilon'$, $\pm\tau'$		n, $\pm n\upsilon'$, $\pm\tau'$
$4i+1=p$	1, $\pm f$, g	1, $\pm\tau$, $\pm\upsilon$		1, $\pm\tau$, $\pm\upsilon$
$2(4i\pm1)$				1, $\pm\tau$, $\pm\upsilon$

Addenda and Corrigenda.

Addendum. Some explanation is much wanted of the Table of Y_2, Z_2 (on pages 170, 171 of Memoir i., and pages 8, 9 of Memoir ii.); none is given in the Text.

This Table has the Title

$$X_{4n} = Y_2^2 + \left(\frac{-1}{n}\right) 2nzZ_2^2, \quad \left[\text{where } \left(\frac{-1}{n}\right) \text{ means } (-1)^{\frac{1}{2}(n-1)}\right],$$

and seems to be a sequel to the preceding Table of Y_1, Z_1, but the explanations given for that Table do not apply; it seems however to be a Table of Y_2, Z_2 in the formula

$$X_{4n} = Y_2^2 - 2nzZ_2^2 \text{ [the connecting sign always } -\text{]}.$$

Only one half of the coefficients b_r, c_r are printed, so that the following directions are needed—

Y_2 is of degree μ, with $\mu+1$ coefficients,
Z_2 is of degree $\mu-1$ with $\mu-1$ coefficients.

Y_2, Z_2 are symmetric functions of $(z, 1)$, so that the omitted coefficients may be supplied by the Rule

$$b_r = b_{\mu-r}, \quad c_r = c_{\mu-r}.$$

* The determinant of a 2ic form is said to be *Idoneal* when it has *only one class* in each of its genera.

Corrigenda in the Text.

Memoir	*Page*	*Line*	*For*	*Read*
i.	167	11 up	1, 17, 21, ..., 21, 17, 1;	1, 7, 21, ..., 21, 7, 1.
i.	168	2	z_1	Z_1
ii.	5	1	1, 17, 21, ..., 21, 17, 1;	1, 7, 21, ..., 21, 7, 1.
ii.	5	13	z_1	Z_1
i. ii.	165 2	The rule for signs of b_r, $b_{\mu-r}$ is incorrect when $n=4i+3$ is *composite:* in that case $b_r=+b_{\mu-r}$.		

Corrigenda in the Tables.*

Memoir	*Tab. of*	*Page*	*Col.*	*Line of n*	*For*	*Read*
i.	Y, Z	165	Y	.3	$\dots -7-2]$	$\dots -7-4]$
			Z	11	$[1+3]$	$[1+0]$
			Z	21	$[1+1+1]$	$[1-1+1]$
			Z	19	$[1+1-1-2]$	$[1+0-1+1]$
	Y_1, Z_1	168	Z_1	7	$[1+1]$	$[1-1]$
			Z_1	23	$[\dots 1+7]$	$[\dots 1-7]$
			Y_1	33	$\dots -3^2-19]$	$\dots -3^2-59]$
			Y_1	29	$[1+15+33+15+\dots$	$[1+15+33+13+15+\dots$
			Y_1	41	$[1+21+57+\dots$	$[1+21+67+\dots$
			Z_1 Z_1	41 69	Interchange the lines of $n=41$, 69	
ii.	Y, Z	2	Z	11	$[1+3]$	$[1+0]$
			Z	19	$[1+1-1-2]$	$[1+0-1+1]$
			Z	23	$\dots -1-4]$	$\dots -1-2]$
	Y_1, Z_1	5	Z_1	7	$[1+1]$	$[1-1]$
			Z_1	23	$[\dots +1+7]$	$[\dots +1-7]$
			Y_1	29	$[1+15+33+15+\dots$	$[1+15+33+13+15+\dots$
iii.	Y, Z	6	Y	22	$+x^5y^5$	$+11x^5y^5$
			Y	33	$-19x^5y^5+$	$-59x^5y^5-$
			Y	29	$+15x^{11}y^3$	$+13x^{11}y^3$

X. Lloyd Tanner, H. W. *Complex Primes formed with the Fifth Roots of Unity*, pub. in *Proc. Lond. Math. Soc.*, vol. xxiv., 1892–3, pages 223–262.

The Table on pages 256–262 gives for every prime (p) of forms $p=10\varpi+1$ up to the limit $p>10000$ (save one missing), and for three higher primes, the following data—

1°. The elements (X, Y) of the 2ic partition $p=\frac{1}{4}(X^2-Y^2)$; with X, Y minima, X is always $+$; the sign of Y is fixed in a way explained in Art. 38.

* Some of these Errata were published in the Author's paper on *Factorisation of* $Y^Y \mp 1$), &c., in the *Messenger of Mathematics*, vol. xlv., 1915, page 75; but some fresh ones have been found since then.

The Table of the Y, Z of formula (1) alone has been compared with the following Works, wherein the values of Y, Z are given at full length—

1. G. B. Mathews's *Theory of Numbers*, Part I., 1892, page 218, complete for $n=p$ (a prime) up to $n=31$
2. H. Holden's papers in the *Messenger of Mathematics*, vol. xxxv., 1906, pages 79, 80, and vol. xxxvii. 1907–8, pages 136, 137, which give the Y, Z for many values of n up to $n=57$, including $n=43, 51, 57$ (not given in Lucas's Table).

2°. The Residue of the product $3XY$ (mod. 5).

3°. The coordinates (a_0, a_1, &c.) of the "simplest" quintic prime factor $a\omega = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + a_4\omega^4$ of p, where ω is an imaginary root of $\omega^5 - 1 = 0$

4°. The coordinates (b_0, b_1, &c.) of the "simplest primary" quintic prime factor $b\omega = b_0 + b_1\omega + b_2\omega^2 + b_3\omega^3 + b_4\omega^4$ of p.

5°. The coordinates (q_0, q_1, &c.) of the "reciprocal" quintic factors $q\omega = q_0 + q_1\omega + q_2\omega^2 + q_3\omega^3 + q_4\omega^4$ of p; for all primes $p > 1000$ up to 10000 (except one missed).

5°*a*. For primes $p < 1000$ the values of the "simplest" and "simplest primary" quintic factors of p of Reuschle's Tables (described above) in terms of the above $a\omega$, $b\omega$.

6°. The value of $Q = q_1 - q_2 - q_3 + q_4$ for each prime p.

Full description of this Table is given on pages 254, 255 of the Memoir. Note that negative numbers are indicated by placing a dot over each digit of the number (see below).

Corrigenda.

Page 257. In the reciprocal factor of $p = 2161$; for $q_3 = 14$, read 24.

Page 258. The prime $p = 3371$ is missing. Insert* the line

p	X, Y	$3XY$	*Simplest*	*Primary*	*Reciprocal*	Q
3371	127, 23	2	3, 1, 1, 2, 6	0, 5, 4, 2, 8	6, 8, 46, 15, 16	55

XI. Roberts, C. A. *Table of the Square Roots of the Prime Numbers of the forms* $4m + 1$ *less than* 10000 *expressed as Periodic Continued Fractions*, pub. in the *Mathematical Magazine*, vol. ii. (part 7 of 1892), Washington, 1893, pages 105—120.

The Table of 15 quarto pages (106—120) shows† one half of the complete cycle of $(2n + 1)$ denominators of the continued fraction development of primes (N) of form $N = 4m + 1$; and records also the number $(2n + 1)$ of denominators in the full cycle. It extends from $N = 5$ to 10500 (although the limit is stated in the Table to be under 10000).

Corrigenda.

Page	N	*Col.*	*For*	*Read*
107	1553	$2n+1$	17	15
108	1777	,,	61	43

Page	N	*Cycle of denominators* (d).
107	1533	The first 6 are correct; after $d = 9$, read only (3, 3).
108	1777	The first 19 are correct; after $d = 27$, read only 2, (1, 1).

* These data were supplied by the late Mr. Lloyd Tanner himself.

† The same as the upper line for each number (N) in Degen's and Bickmore's Tables.

XII. Bickmore. Chas. E. *Tables connected with the Pellian Equation*, pub. in *Brit. Assocn. Report* of 1893, pages 73—120.

This Work contains two Tables—

Table I. (pages 75—81) contains the Least Solutions (x, y) of one or other of the two equations

$$y^2 - ax^2 = \pm A, \quad [a \neq \square] \text{ up to } a = 99, \quad A \not> 2\sqrt{a}$$

for the whole of the "auxiliary numbers" $(\pm A)$ of the lower line of Degen's Table (being those for which solution is possible).

Table II. (pages 82—119) gives the elements $(d, \pm A)$ of the continued fraction development of $\sqrt{a}$ $(a \neq \square)$, from $a = \dagger 1001$ to 1500, arranged in two lines on the same pattern as in Degen's Table, and also the solution (x, y) of one‡ or other of the two Pellian Equations, viz.

Of $y^2 - ax^2 = -1$ (when possible) $\}$ $[a \neq \square]$
Of ‡$y^2 - ax^2 = +1$ (when $y^2 - ax^2 \neq -1$) $\}$ $a = \dagger 1001$ to 1500.

The former case is distinguished by an asterisk (*) in the Argument (a) column.

The method of computing the solutions (x, y) from the data $(d, \pm A)$ is explained in the Text, pages 73–75, 81.

Corrigenda.

Tab.	*Page*	*Arg.*	*Col.*	*For*	*Read*
I.	76;	29	$y^2 - ax^2$	$+1, -6, +5, -3, +2, -1$;	$+1, -4, +5, -5, +4, -1$
II.	109;	Arg. (a)		$a = 1361$	$a = 1361^*$

XIII. Cunningham, Allan J. C, Lt.-Col. R E. Two Tables (*The Binary Canon* and *Quadratic Partitions*) by the Author are here reviewed.

i. *A Binary Canon*, London, 1900.

This Table is closely similar in arrangement, detail, and scope to Jacobi's Canon (see page 57 above); it differs from the latter only in that the Base 2 is used throughout instead of a primitive root (g) of each prime. With those primes (p) and prime-powers (p^κ) of which p is a primitive root, the Table can be used in the same way as Jacobi's; in other cases its use is more limited.

A short Table of 8 Corrigenda is printed on the back

† Thus this Table is an extension of Degen's Table (which stops at $a = 1000$) from $a = 1001$ to 1500

‡ The omission of the solution of this, the more important equation (with the + sign), whenever $y^2 - ax^2 = -1$ is possible, is a decided inconvenience.

of the Table of Contents. The following *additional* Corrigenda have since been discovered.

Table of Residues.					*Table of Indices.*					*Value of* $(p-1)$			
Pge.	*Tab.* p	*Arg.* x	*For*	*Read*	*Pge.*	*Tab* p, p^κ	*Arg.* R	*For*	*Read*	*Pge.*	p	*For*	*Read*
60	547	435	102	192	122	853	$R=$	300	300				
104	773	699	873	73				210	310	3	41	2^3	$2^3.5$
120	839	315	541	548	147	947	$R=$	910	910	10	139	2.69	2.3.23
127	859	526	776	770				820	920				
					162	289	233	Bla-nk	Stop (.)				

ii. *Quadratic Partitions*, London, 1914.

This Work opens with an "Introduction" of 23 pages, giving a description of the Tables themselves, and of some of their uses.

The Tables themselves cover 264 pages. These give

1°. The factorisation of $(p-1)$ for all primes $(p) < 100000$.

2°. The elements (a, b), (c, d), &c., ..., (t, u) of the 2^{ic} partitions stated of all primes up to the limits stated.

Partition.	*Limits.*
a^2+b^2, c^2+2d^2, A^2+3B^2, $\frac{1}{4}(L^2+27M^2)$	$p \geqslant 100000$
e^2-2f^2	$p \geqslant 25000$
$t^2 \mp Du^2$; [D = 5, 6, 7, 10, 11]	$p \geqslant 10000$
$t^2 \pm Du^2$; [D = 13, 14, 15, 17, 19]	$p \geqslant 1000$
Least solutions of $\tau^2 - Dv^2 = \mp 1$	$D \neq \square^2$ and $\geqslant 100$
Multiple solutions of $\tau^2 - Dv^2 = \mp 1$	$D = 2$ to 20
Least solutions of $\tau^2 - Dv^2 = \pm 2, \pm 8, \pm 16$	$D \geqslant 500$
Least solutions of $\tau^2 - Dv^2 = \pm 4$	$D \geqslant 1000$

A short Table of Corrigenda is printed on the back of the Table of Contents. The following *additional* Corrigenda have since been discovered.

Corrigenda (in the "Introduction").

Page	*Line*	*For*	*Read*
i.	10 up	$\frac{1}{4}(v^2+w^2)$	$\frac{1}{4}(v^2+11w^2)$
ix.	4 up	$(3\tau^2 v + qv^3)$	$(3\tau^2 v + Dv^3)$
xii.	11	be the	be not the
xiii.	3 up	always	sometimes
xiv.	6 up	300024	30024
xvi.	8	9604^2	5.9604^2
xx.	12	$p) = 2521$	$p = 2521)$
,,	15	2^{84}	5^{84}
,,	1 of Tab.	81	80
,,	2 ,,	612	611
,,	3 ,,	| 4807 |...| 2373 |...| 1180 |	| 4806 |...| 2374 |...| 1181 |
xxi.	16	(A, B), $(p-1)$	(A, B), (L, M), $(p-1)$
,,	5 up	causes	cause

Corrigenda (in the Tables).

1°. Corrigenda in $(p-1)$ column; [Arg. p].
$p=87481$; read $p-1=8.27.81\ 5$.
$p=96661$; read $p-1=4.27.5.179$.

2°. Corrigenda in the 2^{le} parts a, b; [Arg. p].
$p=45289, 55633, 70289, 77549, 79609, 80809, 95101$;
Interchange the printed a, b; [a should be *odd*, b *even*].

3°. Corrigenda in the 2^{le} parts A, B; L, M; [Arg. p].
$p=60169$; read A, B = 37, 140; L, M = 383, 59.

XIV. Whitford, E. E. *The Pell Equation*, New York, 1912.

This Work is a Monograph (190 pages) on the Pell Equation, containing its history (pages 1—76), methods of solution (pages 95—101), and an extensive bibliography (pages 113—161).

There are two Tables—

Table I. (pages 102—112) gives the least solution (x, y) of both the Pellian Equations

$$x^2-Ay^2=-1, \text{ and } x^2-Ay^2=+1, \ [A \neq \square] \text{ from } {}^*A=1501 \text{ to } 1700.$$

The solutions are not distinguished by any special mark; but those of the former (when existent) are placed above those of the latter, and are always *so much smaller* than those of the latter that they cannot be mistaken.

Table II. in the Appendix (pages 163—190) contains the elements of the continued fraction development of $\sqrt{(A)}$ from $A={}^*1501$ to 2012, $[A \neq \square]$, on the same patterns as in Degen's Table (see page 56, above).

Corrigenda.

Page	*Number*	*Cycle of Denominators.*	
175	$N=1733$	For 1, 1, 1, 2, 2, 4, ...	Read 1, 1, 1, 2, 3, 4, ...

* Thus these two Tables are a continuation of Bickmore's Table (No. XII, above), which extends only to 1500.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLIII.

On the asymptotic value of a definite integral, and the coefficient in a power series.

1. I HAVE had occasion recently to work out an asymptotic value for the integral

$$J(p) = \int_0^\infty e^{-x^r + rax} x^p \, dx, \tag{1}$$

where $r > 1$, $a > 0$, and p is large and positive. I state the results in § 2: I have not thought it worth while to give more than an outline of the proof, which involves no novelty of idea*.

In §§ 3—4 I use the results to find an asymptotic formula for the coefficients of the series which represents the function

$$f(x) = \frac{1}{(1-x)^\alpha} \exp\left\{\frac{c}{(1-x)^q}\right\}, \tag{2}$$

where α, c, and q are real and the last two positive. This function is of much interest as being the simplest function which is of infinite order in the circle of convergence of its Taylor's series. Here also I have not troubled to give the details of the proof.

2. The function

$$\phi(x) = e^{-x^r + rax} x^p$$

attains its maximum when x is equal to the positive root x_0 of the equation

$$x^r = ax + \frac{p}{r},$$

the first approximation to which is $(p/r)^{1/r}$. If we put $x = x_0 + \eta$, where η is small, we find that

$$\phi(x) = e^{P - Q\eta^2 + O(\eta^3)},$$

where

$$P = -x_0^r + rax_0 + p \log x_0 = (r-1)\, ax_0 + p \log x_0 - \frac{p}{r}$$

* The principle of the method dates back, I believe, to Laplace.

and

$$Q = \frac{r(r-1)\,x_0^{r}+p}{2x_0}, \sim \tfrac{1}{2}rp\left(\frac{p}{r}\right)^{-2/r}.$$

The arguments usual in dealing with problems of this character show that

$$J(p) \sim e^{P}\sqrt{\left(\frac{\pi}{Q}\right)};$$

and all that is necessary, in order to obtain for $J(p)$ an asymptotic formula of the type $J(p) \sim \psi(p)$, is to approximate to P with an error which is small when p is large.

I find that

$$(3)\quad J(p) \sim \sqrt{\left(\frac{2\pi}{rp}\right)}\left(\frac{p}{r}\right)^{1/r}\exp\left\{\frac{p}{r}\log\frac{p}{r} - \frac{p}{r} + ra\left(\frac{p}{r}\right)^{1/r}\right\}$$

if $r > 2$, and

$$(4)\quad J(p) \sim \sqrt{(\tfrac{1}{2}\pi)}\exp\left\{\tfrac{1}{2}p\log\tfrac{1}{2}p - \tfrac{1}{2}p + 2a\sqrt{(\tfrac{1}{2}p)} + \tfrac{1}{2}a^{2}\right\}$$

if $r = 2$. As r decreases towards 1 the formula becomes more complicated, but may be calculated in any case with sufficient trouble. In all cases we have

$$J(p) = \exp\left\{\frac{p}{r}\log\frac{p}{r} - \frac{p}{r} + ra\left(\frac{p}{r}\right)^{1/r} + o\left(p^{1/r}\right)\right\}.$$

3. Suppose first that α is positive. Then

$$f(x) = \frac{1}{(1-z)^{\alpha}}\exp\left\{\frac{c}{(1-z)^{q}}\right\} = \sum_{0}^{\infty}\frac{c^{k}}{k!}(1-z)^{-qk-\alpha}$$
$$= \sum_{0}^{\infty} A_p z^{p},$$

where

$$A_p = \frac{1}{p!}\sum_{0}^{\infty}\frac{\Gamma(qk+p+\alpha)}{k!\,\Gamma(qk+\alpha)}c^{k}.$$

And we have

$$(5)\qquad A_p = \frac{1}{p!}\int_{0}^{\infty} e^{-u}u^{p+\alpha-1}F(cu^{q})\,du,$$

where

$$F(x) = \sum_{0}^{\infty}\frac{x^{k}}{k!\,\Gamma(qk+\alpha)} = \sum_{0}^{\infty} C_k x^{k},$$

say. A simple approximation depending on Stirling's Theorem shows that

$$C_k^{1/k} = e^{q+1-q\log q}\, k^{-q-1}\left\{1 + A\,\frac{\log k}{k} + \frac{B}{k} + o\left(\frac{1}{k}\right)\right\},$$

where

$$A = -\alpha, \quad B = -\log 2\pi - (\alpha - \tfrac{1}{2})\log q.$$

It follows, from a general theorem of Lindelöf*, that

$$(6)\quad F(x) \sim \frac{1}{\sqrt{\{2\pi\,(q+1)\}}}\,(qx)^{\frac{\frac{1}{2}-\alpha}{q+1}} \exp\left\{(q+1)\,q^{-\frac{q}{q+1}}\,x^{\frac{1}{q+1}}\right\}$$

as $x \to \infty$.

The asymptotic value of the coefficient A_p may be found by substituting for $F(cu^q)$, in (5), its asymptotic value given by (6), and writing

$$u = w^r,$$

where $r = (q+1)/q$. We thus obtain a definite integral of the type considered in § 2. If $q < 1$, $r > 2$, and we can apply (3). In this case we find

$$(7)\quad A_p \sim \frac{1}{\sqrt{\{2\pi\,(q+1)\}}}\,(qc)^{\frac{\frac{1}{2}-\alpha}{q+1}}\, p^{\frac{\frac{1}{2}q+\alpha}{q+1}-1} \exp\left\{\frac{q+1}{q}\,(qc)^{\frac{1}{q+1}}\, p^{\frac{q}{q+1}}\right\}.$$

If $q = 1$, $r = 2$, and we must use (4). In this case

$$(8)\qquad A_p \sim \frac{1}{2\sqrt{\pi}}\, c^{\frac{1}{4}-\frac{1}{2}\alpha}\, e^{\frac{1}{2}c}\, p^{-\frac{3}{4}+\frac{1}{2}\alpha}\, e^{2\sqrt{(cp)}}.$$

If $q > 1$ the formula becomes more complicated, but may be calculated in any case with sufficient trouble. In all cases

$$(9)\qquad \log A_p \sim \frac{q+1}{q}\,(qc)^{\frac{1}{q+1}}\, p^{\frac{q}{q+1}}.$$

In proving these formulæ we have supposed α positive. This restriction is however quite unnecessary, and may be removed either by an appropriate modification of the formula (5), or by successive multiplications by $1 - x$.

4. It would not be difficult to remove the restriction that α is *real*. A more interesting question is that of extending the formulæ to complex values of c. I have no doubt, for example, nor do I think it would be difficult to prove, that

* E. Lindelöf, 'Sur la détermination de la croissance des fonctions entières définies par un développement de Taylor', *Bulletin des Sciences Mathématiques*, vol. xxvii. (1903), pp. 213–226.

(8) holds whenever the real part of $\sqrt{c}$ is positive, that is to say for all values of c other than real negative values.

The case in which e is real and negative, which is intrinsically more difficult, has been considered by Fejér*. Fejér takes $c=-1$, and proves that

$$\frac{1}{(1-z)^{\alpha}}\exp\left(-\frac{1}{1-z}\right)=\Sigma A_p z^p$$

where

$$A_p \sim \frac{1}{\sqrt{(e\pi)}}\, n^{-\frac{3}{4}+\frac{1}{2}\alpha}\sin\{2\sqrt{n}+(\tfrac{3}{4}-\tfrac{1}{2}\alpha)\pi\}.$$

A SERIES FOR EULER'S CONSTANT γ.

By *S. Ramanujan.*

1. In a paper recently published in this journal (vol. xliv., pp. 1–10), Dr. Glaisher proves a number of formulæ of the type

$$\gamma=1-2\left(\frac{S_3}{3.4}+\frac{S_5}{5.6}+\frac{S_7}{7.8}+\ldots\right),$$

where $$S_n=1^{-n}+2^{-n}+3^{-n}+4^{-n}+\ldots,$$

and conjectures the existence of a general formula

$$\gamma=\lambda_r-(r+1)(r+2)\ldots(2r)$$
$$\times\left\{\frac{S_3}{3(r+3)(r+4)\ldots(2r+2)}+\frac{S_5}{5(r+5)(r+6)\ldots(2r+4)}+\ldots\right\},$$

where λ_r is a rational number. I propose now to prove the general formula of which Dr. Glaisher's are particular cases: this formula is itself a particular case of still more general formulæ.

2. Let r and t be any two positive numbers. Then

$$(1)\quad \int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx=\int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(x)\,dx$$
$$=\int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\,dx-\int_0^1 x^{t-1}(1-x)^{r-1}\log x\,dx.$$

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* 'Sur une méthode de M. Darboux', *Comptes Rendus*, 30 Nov. 1908; and 'Asymptotikus értékek meghatározásáról', *Mathematikai és természettudományi értesitö*, vol. xxvii. (1909), pp. 1–33.

But

$$(2)\quad \int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma(1-x)\,dx$$

$$=\int_0^1 x^{r-1}(1-x)^{t-1}\left\{\gamma x+S_2\frac{x^2}{2}+S_3\frac{x^3}{3}+\ldots\right\}dx$$

$$=\frac{\Gamma(1+r)\Gamma(t)}{\Gamma(1+r+t)}\gamma+\frac{\Gamma(2+r)\Gamma(t)}{\Gamma(2+r+t)}\frac{S_2}{2}+\frac{\Gamma(3+r)\Gamma(t)}{\Gamma(3+r+t)}\frac{S_3}{3}+\ldots,$$

Similarly

$$(2')\qquad \int_0^1 x^{t-1}(1-x)^{r-1}\log\Gamma(1+x)\,dx$$

$$=-\frac{\Gamma(1+t)\Gamma(r)}{\Gamma(1+r+t)}\gamma+\frac{\Gamma(2+t)\Gamma(r)}{\Gamma(2+r+t)}\frac{S_2}{2}-\frac{\Gamma(3+t)\Gamma(r)}{\Gamma(3+r+t)}\frac{S_3}{3}+\ldots.$$

And also

$$(3)\quad \int_0^1 x^{t-1}(1-x)^{r-1}\log x\,dx=\frac{d}{dt}\int_0^1 x^{t-1}(1-x)^{r-1}dx=\frac{d}{dt}\left\{\frac{\Gamma(t)\Gamma(r)}{\Gamma(r+t)}\right\}$$

$$=\frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\left\{\frac{\Gamma'(t)}{\Gamma(t)}-\frac{\Gamma'(r+t)}{\Gamma(r+t)}\right\}=-\frac{\Gamma(r)\Gamma(t)}{\Gamma(r+t)}\int_0^1 x^{t-1}\,\frac{1-x^r}{1-x}\,dx.$$

It follows from (1)—(3) that, if r and t are positive, then

$$(4)\quad \frac{r}{1\,(r+t)}\gamma+\frac{r\,(r+1)}{2\,(r+t)\,(r+t+1)}S_2$$

$$+\frac{r\,(r+1)\,(r+2)}{3\,(r+t)\,(r+t+1)\,(r+t+2)}S_3+\ldots$$

$$+\frac{t}{1\,(r+t)}\gamma-\frac{t\,(t+1)}{2\,(r+t)\,(r+t+1)}S_2$$

$$+\frac{t\,(t+1)\,(t+2)}{3\,(r+t)\,(r+t+1)\,(r+t+2)}S_3-\ldots$$

$$=\int_0^1\frac{x^{t-1}\,(1-x^r)}{1-x}\,dx.$$

Now, interchanging r and t in (4), and taking the sum and

the difference of the two results, we see that, if r and t are positive, then

$$(5)\quad \frac{r+t}{1\,(r+t)}\gamma+\frac{r\,(r+1)\,(r+2)+t\,(t+1)\,(t+2)}{3\,(r+t)\,(r+t+1)\,(r+t+2)}S_3+\ldots$$
$$=\tfrac{1}{2}\int_0^1\frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx;$$

and

$$(6)\quad \frac{r\,(r+1)-t\,(t+1)}{2\,(r+t)\,(r+t+1)}S_2$$
$$+\frac{r\,(r+1)\,(r+2)\,(r+3)-t\,(t+1)\,(t+2)\,(t+3)}{4\,(r+t)\,(r+t+1)\,(r+t+2)\,(r+t+3)}S_4+\ldots$$
$$=\tfrac{1}{2}\int_0^1\frac{x^{t-1}-x^{r-1}}{1-x}\,dx.$$

The right-hand sides of (5) and (6) can be expressed in finite terms if r and t are rational. If, in particular, r and t are integers, then

$$\int_0^1\frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx=\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\ldots+\frac{1}{r+t-1}$$
$$+\frac{1}{t}+\frac{1}{t+1}+\frac{1}{t+2}+\ldots+\frac{1}{r+t-1};$$

and

$$\int_0^1\frac{x^{t-1}-x^{r-1}}{1-x}\,dx=\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{r-1}\right)$$
$$-\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{t-1}\right).$$

3. Let us now suppose that $t=r$ in (6). Then it is clear that

$$(7)\quad \gamma+\frac{(r+1)\,(r+2)}{3(2r+1)\,(2r+2)}S_3+\frac{(r+1)\,(r+2)\,(r+3)\,(r+4)}{5\,(2r+1)\,(2r+2)\,(2r+3)\,(2r+4)}S_5$$
$$+\ldots=\int_0^1\frac{x^{r-1}\,(1-x^r)}{1-x}\,dx=\int_0^1\frac{1+x^{2r-1}}{1+x}\,dx,$$

if $r>0$. If we suppose, in (7), that r is an integer, we obtain the formula conjectured by Dr. Glaisher, the value of λ_r being

$$\int_0^1\frac{1+x^{2r-1}}{1+x}\,dx=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots+\frac{1}{2r-1}.$$

Again, dividing both sides in (6) by $r-t$ and making $t \to r$, we see that, if $r > 0$, then

$$(8)\quad \frac{r+1}{2(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)S_2$$

$$+\frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)S_4+\dots$$

$$=-\int_0^1 \frac{x^{r-1}\log x}{1-x}\,dx=\frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\frac{1}{(r+3)^2}+\dots.$$

Thus for example we have

$$\frac{\pi^2}{12}=(1+\tfrac{1}{2})\frac{S_2}{2.3}+(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})\frac{S_4}{4.5}$$

$$+(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6})\frac{S_6}{6.7}+\dots.$$

4. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\log\Gamma\left(1-\frac{x}{2}\right)dx,$$

and proceed as in § 2, we can show that, if r and t are positive, then

$$(9)\quad \frac{r}{1(r+t)}S_1'+\frac{r(r+1)}{2(r+t)(r+t+1)}S_2'$$

$$+\frac{r(r+1)(r+2)}{3(r+t)(r+t+1)(r+t+2)}S_3'+\dots$$

$$-\frac{t}{1(r+t)}S_1'+\frac{t(t+1)}{2(r+t)(r+t+1)}S_2'$$

$$-\frac{t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3'+\dots$$

$$=\int_0^1 \frac{x^{t-1}(1-x^r)}{1-x}\,dx-\log\frac{\pi}{2},$$

where $$S_n'=1^{-n}-2^{-n}+3^{-n}-4^{-n}+\dots.$$

From (9) we can easily deduce that, if r and t are positive, then

$$(10)\quad \frac{r(r+1)+t(t+1)}{2(r+t)(r+t+1)}S_2' + \frac{r(r+1)(r+2)(r+3)+t(t+1)(t+2)(t+3)}{4(r+t)(r+t+1)(r+t+2)(r+t+3)}S_4' + \dots = \tfrac{1}{2}\int_0^1 \frac{x^{r-1}+x^{t-1}-2x^{r+t-1}}{1-x}\,dx - \log\frac{\pi}{2};$$

and

$$(11)\quad \frac{r-t}{1(r+t)}S_1' + \frac{r(r+1)(r+2)-t(t+1)(t+2)}{3(r+t)(r+t+1)(r+t+2)}S_3' + \dots = \tfrac{1}{2}\int_0^1 \frac{x^{t-1}-x^{r-1}}{1-x}\,dx.$$

As particular cases of (10) and (11), we have

$$(12)\quad \log\frac{\pi}{2} + \frac{r+1}{2(2r+1)}S_2' + \frac{(r+1)(r+2)(r+3)}{4(2r+1)(2r+2)(2r+3)}S_4' + \dots = \int_0^1 \frac{1+x^{2r-1}}{1+x}\,dx;$$

and

$$(13)\quad \frac{1}{r}S_1' + \frac{(r+1)(r+2)}{3(2r+1)(2r+2)}\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}\right)S_3' + \frac{(r+1)(r+2)(r+3)(r+4)}{5(2r+1)(2r+2)(2r+3)(2r+4)} \times \left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}+\frac{1}{r+4}\right)S_5' + \dots = \frac{1}{r^2}+\frac{1}{(r+1)^2}+\frac{1}{(r+2)^2}+\dots,$$

provided that $r>0$. Thus for example we have

$$1 = \log\frac{\pi}{2} + 2\left(\frac{S_2'}{2.3}+\frac{S_4'}{4.5}+\frac{S_6'}{6.7}+\dots\right);$$

$$\frac{\pi^2}{12} = \frac{S_1'}{1.2} + \frac{S_3'}{3.4}\left(1+\tfrac{1}{2}+\tfrac{1}{3}\right) + \frac{S_5'}{5.6}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}\right)+\dots.$$

5. The preceding results may be generalised as follows. Let $\zeta(s, x)$ denote the function represented by the series

$$x^{-s}+(x+1)^{-s}+(x+2)^{-s}+(x+3)^{-s}+\ldots \quad (x>0)$$

and its analytical continuations, so that $\zeta(s, 1)=\zeta(s)$ and $\zeta(s, \frac{1}{2})=(2^s-1)\zeta(s)$, $\zeta(s)$ being the Riemann ζ-function. Then

$$(14)\quad \int_0^1 x^{r-1}(1-x)^{t-1}\zeta(s, 1-x)\,dx=\int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s, x)\,dx$$

$$=\int_0^1 x^{t-1}(1-x)^{r-1}\zeta(s, 1+x)\,dx+\int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx,$$

provided that r and t are positive. But we know that, if $|x|<1$, then

$$(15)\quad \zeta(s, 1-x)=\zeta(s)+\frac{s}{1!}\zeta(s+1)x+\frac{s(s+1)}{2!}\zeta(s+2)x^2+\ldots;$$

and that

$$(16)\qquad \int_0^1 x^{t-s-1}(1-x)^{r-1}\,dx=\frac{\Gamma(t-s)\Gamma(r)}{\Gamma(r-s+t)},$$

provided that $t>s$. It follows from (14)—(16) that, if r and t are positive and $t>s$, then

$$(17)\quad \left\{\zeta(s)+\frac{s}{1!}\,\frac{r}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\,\frac{r(r+1)}{(r+t)(r+t+1)}\zeta(s+2)+\ldots\right\}$$

$$-\left\{\zeta(s)-\frac{s}{1!}\,\frac{t}{r+t}\zeta(s+1)+\frac{s(s+1)}{2!}\,\frac{t(t+1)}{(r+t)(r+t+1)}\zeta(s+2)-\ldots\right\}$$

$$=\frac{\Gamma(r+t)\Gamma(t-s)}{\Gamma(t)\Gamma(r-s+t)}.$$

As particular cases of (17), we have

$$(18)\quad \frac{s}{1!}\,\frac{r+t}{r+t}\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\,\frac{r(r+1)(r+2)+t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\zeta(s+3)+\ldots$$

$$=\frac{1}{2}\,\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}+\frac{\Gamma(r-s)}{\Gamma(r)}\right\},$$

and

$$(19)\quad \frac{s(s+1)}{2!}\,\frac{r(r+1)-t(t+1)}{(r+t)(r+t+1)}\,\zeta(s+2)$$

$$+\frac{s(s+1)(s+2)(s+3)}{4!}\,\frac{r(r+1)(r+2)(r+3)-t(t+1)(t+2)(t+3)}{(r+t)(r+t+1)(r+t+2)(r+t+3)}\,\zeta(s+4)+\dots$$

$$=\frac{1}{2}\,\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)}-\frac{\Gamma(r-s)}{\Gamma(r)}\right\},$$

provided that r and t are positive and greater than s. From (18) and (19) we deduce that, if r is positive and greater than s, then

$$(20)\quad \frac{s}{1!}\,\frac{r}{2r}\,\zeta(s+1)+\frac{s(s+1)(s+2)}{3!}\,\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\,\zeta(s+3)+\dots$$

$$=\frac{1}{2}\,\frac{\Gamma(2r)\,\Gamma(r-s)}{\Gamma(r)\,\Gamma(2r-s)},$$

and

$$(21)\quad \frac{s(s+1)}{2!}\,\frac{r(r+1)}{2r(2r+1)}\left(\frac{1}{r}+\frac{1}{r+1}\right)\zeta(s+2)$$

$$+\frac{s(s+1)(s+2)(s+3)}{4!}\,\frac{r(r+1)(r+2)(r+3)}{2r(2r+1)(2r+2)(2r+3)}$$

$$\times\left(\frac{1}{r}+\frac{1}{r+1}+\frac{1}{r+2}+\frac{1}{r+3}\right)\zeta(s+4)+\dots$$

$$=\frac{1}{2}\,\frac{\Gamma(2r)\,\Gamma(r-s)}{\Gamma(r)\,\Gamma(2r-s)}\int_0^1\frac{x^{r-s-1}(1-x^s)}{1-x}\,dx.$$

6. If we start with the integral

$$\int_0^1 x^{r-1}(1-x)^{t-1}\,\zeta\left(s,\ 1-\frac{x}{2}\right)dx,$$

and proceed as in § 5, we can show that, if r and t are positive and $t>s$, then

$$(22)\quad \zeta_1(s)+\frac{s}{1!}\,\frac{r}{r+t}\,\zeta_1(s+1)+\frac{s(s+1)}{2!}\,\frac{r(r+1)}{(r+t)(r+t+1)}\,\zeta_1(s+2)+\dots$$

$$+\zeta_1(s)-\frac{s}{1!}\,\frac{t}{r+t}\,\zeta_1(s+1)+\frac{s(s+1)}{2!}\,\frac{t(t+1)}{(r+t)(r+t+1)}\,\zeta_1(s+2)-\dots$$

$$=\frac{\Gamma(r+t)\,\Gamma(t-s)}{\Gamma(t)\,\Gamma(r-s+t)},$$

where $\zeta_1(s)$ is the function represented by the series

$$1^{-s} - 2^{-s} + 3^{-s} - 4^{-s} + \ldots$$

and its analytical continuations. From (22) we deduce that, if r and t are positive and greater than s, then

$$(23)\quad (1+1)\,\zeta_1(s) + \frac{s(s+1)}{2!}\,\frac{r(r+1)+t(t+1)}{(r+t)(r+t+1)}\,\zeta_1(s+2) + \ldots$$

$$= \frac{1}{2}\,\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} + \frac{\Gamma(r-s)}{\Gamma(r)}\right\};$$

and

$$(24)\quad \frac{s}{1!}\,\frac{r-t}{r+t}\,\zeta_1(s+1)$$

$$+ \frac{s(s+1)(s+2)}{3!}\,\frac{r(r+1)(r+2)-t(t+1)(t+2)}{(r+t)(r+t+1)(r+t+2)}\,\zeta_1(s+3) + \ldots$$

$$= \frac{1}{2}\,\frac{\Gamma(r+t)}{\Gamma(r-s+t)}\left\{\frac{\Gamma(t-s)}{\Gamma(t)} - \frac{\Gamma(r-s)}{\Gamma(r)}\right\}.$$

As particular cases of (23) and (24), we have

$$(25)\quad \zeta_1(s) + \frac{s(s+1)}{2!}\,\frac{r(r+1)}{2r(2r+1)}\,\zeta_1(s+2) + \ldots = \frac{1}{2}\,\frac{\Gamma(2r)\,\Gamma(r-s)}{\Gamma(r)\,\Gamma(2r-s)},$$

and

$$(26)\quad \frac{s}{1!}\,\frac{r}{2r}\,\frac{1}{r}\,\zeta_1(s+1)$$

$$+ \frac{s(s+1)(s+2)}{3!}\,\frac{r(r+1)(r+2)}{2r(2r+1)(2r+2)}\left(\frac{1}{r} + \frac{1}{r+1} + \frac{1}{r+2}\right)\zeta_1(s+3) + \ldots$$

$$= \frac{1}{2}\,\frac{\Gamma(2r)\,\Gamma(r-s)}{\Gamma(r)\,\Gamma(2r-s)}\int_0^1 \frac{x^{r-s-1}(1-x^s)}{1-x}\,dx,$$

provided that r is positive and greater than s.

ON PLANE QUARTIC CURVES WITH A NODE AND A CUSP.

By *Harold Hilton.*

§ 1. The theory of plane quartic curves with a node or a cusp has been discussed by various authors*; and quartic curves with two nodes or two cusps are familiar in the form of bicircular quartics. But the case of quartic curves with a node and a cusp appears to have been somewhat neglected. We shall summarize in this paper some of their more interesting properties.

I have to thank Miss R. E. Colomb for kindly verifying many of the results in this paper.

§ 2. In this section we shall explain the notation employed in this paper.

The cusp is C, the node is A. The harmonic conjugate of the line AC with respect to the tangents at the node A meets the tangent at the cusp C in B.

The four tangents from C touch at C_1, C_2, C_3, C_4; and the three tangents from A touch at A_1, A_2, A_3. The tangents at A meet the curve again at H_1, H_2, and the line H_1H_2 meets the curve again at K_1, K_2. The tangent at C meets the curve again at L, and the tangent at L meets the curve again in G_1, G_2. The line AB meets the curve again at N_1, N_2.

The curve has four bitangents which we denote by b_1, b_2, b_3, b_4.

Taking ABC as triangle of reference for homogeneous coordinates, the coefficients of

$$z^4,\ z^3x,\ z^3y,\ z^2y^2,\ z^2xy,\ x^4,\ x^3y,\ x^3z,\ x^2yz$$

in the equation of the quartic must be zero. Hence, taking suitable homogeneous coordinates, the equation of the quartic is of the form

$$z^2x^2+2zy^2(y+mx)+y^2(px^2+2qxy+ry^2)=0.$$

Write now

$$y^2(y+mx)^2-x^2(px^2+2qxy+ry^2)\equiv(y+ax)(y+bx)(y+cx)(y+dx);$$

* See, for instance, Richmond, *Quarterly Journal of Mathematics*, vol. xxvi. (1893), p. 5; *Proc. London Math. Soc.*, vol. xxxiii. (1901), p. 218.

so that

$$a+b+c+d=2m,$$
$$bc+ca+ab+ad+bd+cd=m^2-r,$$
$$bcd+acd+abd+abc=-2q,$$
$$abcd=-p.$$

The equation of the quartic becomes

$$(zx^2+y^3+my^2x)^2=y^2(y+ax)(y+bx)(y+cx)(y+dx).$$

Hence the tangents from the cusp C are

$$y+ax=0,\quad y+bx=0,\quad y+cx=0,\quad y+dx=0.$$

The point C_1 is $(-1,\ a,\ ma^2-a^3)$, &c.

The tangents from the node A are

$$2z=(ad+bc)\,y,\quad 2z=(bd+ca)\,y,\quad 2z=(cd+ab)\,y.$$

For the first of these is

$$2(zx^2+y^3+my^2x)=y(y+ax)(y+dx)+y(y+bx)(y+cx),$$

meeting the quartic where

$$y^2\{(a+d-b-c)\,y+(ad-bc)\,x\}^2=0.$$

The point A_1 is $(2b+2c-2a-2d,\ 2ad-2bc,\ a^2d^2-b^2c^2)$, &c.

The equations of the bitangents are

$$2z-(cd+db+bc)\,y-bcdx=0,\ \&c.,$$

i.e.

$$2az+a\{r-(m-a)^2\}\,y+px=0,\ \&c.$$

For this equation may be written

$$2(zx^2+y^3+my^2x)=y^2(y+ax)+(y+bx)(y+cx)(y+dx),$$

and represents a line meeting the quartic where

$$\{y^2(y+ax)-(y+bx)(y+cx)(y+dx)\}^2=0,$$

or

$$\{px^2+a[r-(m-a)^2]xy-2a(m-a)y^2\}^2=0.$$

§ 3. We now enumerate properties of the quartic with a brief indication of their proof.

The conic $A_1A_2A_3K_1K_2$ touches the quartic at C. The tangents at K_1 and K_2 pass through A.

The line H_1H_2 is

$$2(pm^2+q^2)\,x+(qr+2pm)\,y+(2q-rm)\,z=0,$$

and the conic is

$$(pm^2+q^2)\,x^2+(qr+2pm)\,xy+(2q-rm)\,(xz+my^2)=0.$$

The conic $C_1C_2C_3C_4A$ touches CG_1 and CG_2 at G_1 and G_2, and touches AB at A.

The line G_1G_2 is

$$2z+ry+(2q-mr)\,x=0,$$

and the conic is

$$z^2+(2q-mr)\,zx+rzy-(p-2qm+rm^2)\,y^2=0.$$

The conic $CC_1C_2C_3C_4$ passes through N_1 and N_2.

The conic is

$$z\,(y+mx)+(px^2+2qxy+ry^2)=0.$$

The conic ACH_1H_2L touches the quartic at L.

The conic is

$$2mzx+y\,(2qx+ry+2z)=0.$$

The points of contact of the tangents from B lie on a conic through C.

The conic is

$$z\,(3y+2mx)+px^2+3qxy+2ry^2=0.$$

The pencils $C(C_1C_2C_3C_4)$ and $A\,(A_1A_2A_3C)$ have the same cross-ratio.

The line meeting the quartic in an equi-anharmonic range envelopes a curve of the fourth class. The envelope touches the ten inflexional tangents, has a node at A with the same tangents as the quartic, has a cusp at C, and touches CB elsewhere.

The envelope has the tangential equation

$$p^4\nu^4-4pm\,\nu^3\lambda+2\nu^2\{2\,(m^2+3r)\,\lambda^2-6q\,\lambda\mu+p\mu^2\}$$
$$-4\lambda\mu\nu\,(3\lambda-2m\mu)+\mu^4=0.$$

§ 4. The following are some of the properties of the bitangents.

The conic through C, A, C_1 and the points of contact of the bitangent b_1 touches AB at A and CB at C.

The conic is

$$zx+(m-a)\,y^2=0.$$

The conic touching the bitangents and the line AC touches AC at C; and the other tangent from A to the conic is AB.

The conic has the tangential equation

$$2\lambda^2+p\mu\nu=2q\nu\lambda.$$

The tangents at A to the two conics touching the bitangents and passing through A form with the tangents at A to the quartic an involution of which AB and AC are double lines.

The tangents from A to a conic touching the bitangents are of the form

$$(cdy - 2z)\,(aby - 2z) = k\,(ady - 2z)\,(bcy - 2z);$$

and the pairs of tangents at A are $py^2 = -4z^2$, $py^2 = -z^2$. They form a pencil of cross-ratio 9, and are both real or both unreal.

The points of contact of the bitangents b_1 and b_2 and the points C, C_1, C_2 lie on a conic.

The conic is

$$px^2 + \{r - (m-a)(m-b)\}\,y^2 + 2yz + (a+b)\,zx + 2qxy = 0.$$

The cross-ratio of the pencil formed by the tangents from C to the quartic is equal to the cross-ratio of the range in which AC meets the bitangents.

If the bitangents b_1 and b_4 meet at the point B_{14}, while b_2 and b_3 meet at B_{23}, the line $B_{14}B_{23}$ passes through A_1. If the line meets AB in P and AC in Q, Q is a double point of the involution $(B_{14}B_{23}, A_1P)$.

For B_{14} is the point $(-2b-2c,\ 2bc,\ b^2c^2)$, &c.

§ 5. We consider here a few special cases.

A quartic cannot have a cusp and a biflecnode.

If each branch of the quartic at A had an inflexion, we should have $z^2 + py^2$ a factor of $2y^2(mz + qy)$; which is impossible.

If A is a flecnode, the two pairs of tangents from C form an involution with CA, CB. The four bitangents meet the line AC in two pairs of points forming an involution with C, A.

The condition for a flecnode at A (one branch of the quartic at A having an inflexion) is

$$(ad - bc)\,(bd - ca)\,(cd - ab) = 0.$$

If CB passes through A_1, it is a double ray of the involution formed by two pairs of tangents from C to the quartic.

In this case $a + d = b + c$.

If the two bitangents b_1, b_2 meet on CB, two tangents from C to the quartic are harmonic conjugates with respect to CA and CB.

In this case $c + d = 0$.

If the bitangent b_1 touches at P and Q, and CB passes through Q, then P and C_1 lie on AB, while P, C_2, C_3, C_4 are collinear.

In this case $m = a$.

§ 6. We now describe the appearance of the quartic. We may classify the different kinds of quartic with node and cusp by means of the quantities a, b, c, d.

I. None of a, b, c, d real.

The quartic consists of a single circuit (*Zug*), with a cusp and crunode (node with real tangents). The points C_1, C_2, C_3, C_4 are unreal; A_1, A_2, A_3 are real. The four bitangents are unreal. Every line meets the curve in at least two real points. The quartic cannot be projected into a closed curve. In the other classes the quartic meets some line in no real point, and we shall take this line as the line at infinity in the following description.

II. a, d real: b and c conjugate complex quantities.

(i.) $ad > 0$.

The quartic consists of a single closed circuit with a cusp and crunode.

(ii.) $ad < 0$.

The quartic consists of a single closed pear-shaped circuit with a cusp, and of an acnode (node with unreal tangents) outside it.

In both cases C_1, C_4, A_1, b_1, b_4 are real, and C_2, C_3, A_2, A_3, b_2, b_3 unreal.

III. a, b, c, d all real.

(i.) All of a, b, c, d have the same sign.

The quartic consists of a closed circuit with a cusp and a crunode, and of a closed oval.

(ii.) Three of a, b, c, d have the same sign.

The quartic consists of a pear-shaped oval, of a second oval, and an acnode.

(iii.) Two of a, b, c, d are positive and two negative.

The quartic consists of a pear-shaped oval with a cusp, and a figure-of-eight oval with a crunode.

In all three cases C_1, C_2, C_3, C_4, A_1, A_2, A_3, b_1, b_2, b_3, b_4 are real.

The following remarks apply to all the classes.

Suppose we travel from C along CB, in such a direction as to initially travel away from the curve at C, until we reach B (after passing through the point at infinity on BC if necessary). Then the points N_1 and N_2 are real, if and only if we pass over one or three of the tangents from A to the quartic. For

$$4\,(q^2 - pr) \equiv (ad + bc)\,(bd + ca)\,(cd + ab).$$

The points G_1 and G_2 are real, if and only if four, two, or none of the intersections of BC_1, BC_2, BC_3, BC_4 with AC lie between A and C.

For the lines CG_1 and CG_2 are

$$ry + (2q - mr)\,x = \pm 2\,(-p + 2qm - rm^2)^{\frac{1}{2}}\,y,$$

while

$$(m-a)\,(m-b)\,(m-c)\,(m-d) \equiv -p + 2qm - rm^2.$$

Not more than four inflexions are real if A is a crunode, or six, if A is an acnode.

§ 7. The curve shown in fig. 1 has the equation

$$100x^2y^2 + 2y\,(100 + 90x) - 144x^2 + 36x + 469 = 0.$$

We have here

$$a = .6,\ b = -.8,\ c = -1, \qquad d = 3:$$

$$m = .9,\ q = .18,\ p = -1.44,\ r = 4.69.$$

The bitangents are*

$$12x - 10y = 23, \qquad 9x + 10y = -9,$$

$$36x + 50y = -27, \quad 12x - 50y = 7.$$

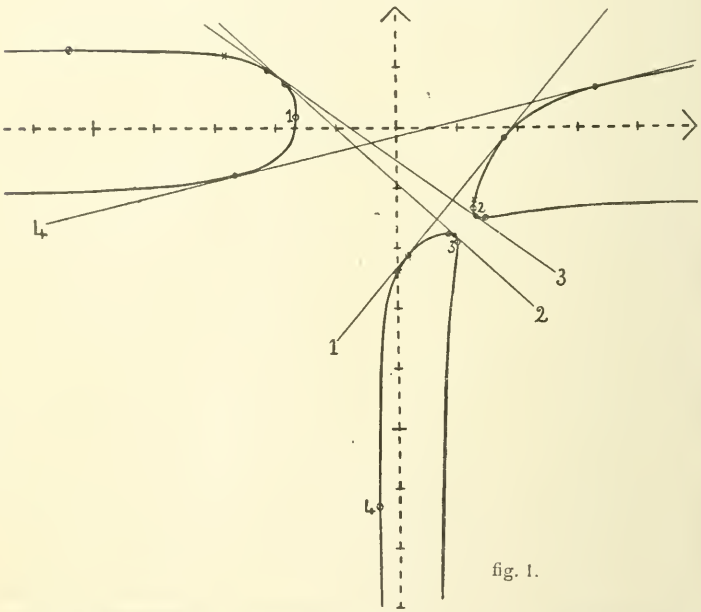

fig. 1.

* It will be noticed that b_1 and b_3 meet on a tangent at A. The condition for this is $4ac = bd$, which happens to be satisfied in this example.

The curve is of the type described in § 6, III. (iii.).

In fig. 1 the bitangents b_1, b_2, b_3, b_4 are denoted by the digits 1, 2, 3, 4; and the points C_1, C_2, C_3, C_4 also by the digits 1, 2, 3, 4. The point B is the origin, A is the point at infinity on the axis of x, and C the point at infinity on the axis of y. Similarly for fig. 2. In fig. 1 H_1, H_2 and L are marked by a cross.

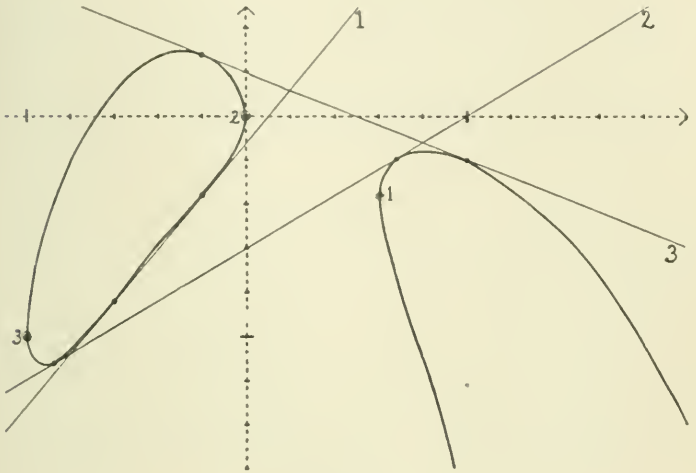

fig. 2.

§ 8. We now give briefly the properties of the quartic in which the node and cusp are united in one singular point, which is usually called a "rhamphoid" cusp.

Taking any coordinates and writing down the first polar curve of the cusp we see that

The points of contact of the three tangents from a rhamphoid cusp of a quartic lie on a conic osculating the quartic at the cusp.

Take the cusp as C, A as any point on the tangent at C, and B as the point of contact of the second tangent from A to this conic. Then, if ABC is the triangle of reference, the equation of the quartic becomes

$$(zy + x^2)^2 = 4y\,(x - ay)\,(x - by)\,(x - cy).$$

By choosing A suitably on the tangent at C we may make b zero if desired.

We call C_1, C_2, C_3 the points of contact of the tangents from C, and we call the three bitangents b_1, b_2, b_3.

The point C_1 is $(a, 1, -a^2)$ and the bitangent b_1 is

$$(b+c-1)\,x+(a-bc)\,y+z=0,$$

touching where

$$x^2-(b+c+1)\,xy+(a+bc)\,y^2=0.$$

We very readily prove that

The intersections with AC of b_1, b_2, b_3 form together with C a range of the same cross-ratio as the pencil $C\,(C_1, C_2, C_3, A)$.

The points of contact of b_1 and C_1 lie on a conic osculating the quartic at C.

The line joining the intersection of b_2 and b_3 to the intersection of b_1 and AC passes through C_1.

The points of contact of b_2 and b_3 lie on a conic through C, C_2, C_3.

The envelope of a line divided in an equianharmonic range by the quartic is a curve of the fourth class with a rhamphoid cusp at C and AC as tangent at C. It touches the inflexional tangents.

§ 9. There are two types of quartic with a rhamphoid cusp C. One has three real tangents from C and three real bitangents: it consists of two ovals, one containing the cusp. The other has one real tangent from C and one real bitangent: it consists of one oval only.

The former type may be illustrated by the curve shown in fig. 2.* Its equation is

$$(5y+x^2)^2=4x\,(x-3)\,(x+5).$$

The tangents from C are $x=3$, $x=0$, $x=-5$. The bitangents are $6x-5y=3$, $3x-5y=15$, $2x+5y=5$.

As an example of the latter type the reader may take

$$(y+x^2)^2=4x\,(x^2-2x+2),$$

with bitangent $x+y=2$.

* It will be noticed that one point of contact of each of b_1 and b_3 are collinear with C. The condition for this is $a+c+2=2b$; which happens to be satisfied in this case.

ON THE EQUATIONS OF EQUILIBRIUM FOR AN ELASTIC ISOTROPIC BODY, WITH OR WITHOUT HEREDITY.

By *C. E. Weatherburn, D.Sc.*

THE present paper shows first how the equations of equilibrium for an elastic isotropic body may be very neatly obtained by vector methods; and then how, by introducing the time-effect, Volterra's integro-differential equations of equilibrium* may be deduced for a body possessing heredity or residual elasticity.

Strain Relations.

If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three mutually perpendicular unit vectors in the directions of the coordinate axes the displacement of the point (x, y, z) may be denoted by

$$\mathbf{D} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w,$$

the six "components of deformation" being

$$a = \frac{\partial u}{\partial x}, \quad b = \frac{\partial v}{\partial y}, \quad c = \frac{\partial w}{\partial z},$$

$$f = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad g = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad h = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

According to the ordinary theory of strains the effect of the deformation at any point of an isotropic body may be represented by the operation of a certain dyadic Ψ; so that if $\mathbf{1}$ be the infinitesimal vector joining two neighbouring particles of the rigid body before the strain, the vector $\mathbf{1}'$ joining the same two particles after the deformation is given by

$$\mathbf{1}' = \Psi \cdot \mathbf{1}.$$

This dyadic may be expressed as the sum† of a self-

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* Cf. *Atti Lincei* (5), t. 18_2 (1909), pp. 585—6; *Acta Math.*, t. 35 (1912), pp. 344—347.

† Cf. *e.g.*, Silberstein, *Vectorial Mechanics*, chap. v. (London, 1913).

conjugate dyadic Ω to be used in direct multiplication, and a vector $\frac{1}{2}\mathsf{e}$ to be used in cross multiplication; that is

$$\Psi \cdot \mathsf{l} = \Omega \cdot \mathsf{l} + \tfrac{1}{2}\mathsf{e} \times \mathsf{l}.$$

The vector e is the curl of the displacement $\mathbf{D}$ at the point considered, while the dyadic Ω is the operator

$$\Omega = \mathsf{ia} + \mathsf{jb} + \mathsf{kc}$$

in which the vectors a, b, c are given by

$$(1) \qquad \begin{cases} \mathsf{a} = a\mathsf{i} + \frac{1}{2}h\mathsf{j} + \frac{1}{2}g\mathsf{k}, \\ \mathsf{b} = \frac{1}{2}h\mathsf{i} + b\mathsf{j} + \frac{1}{2}f\mathsf{k}, \\ \mathsf{c} = \frac{1}{2}g\mathsf{i} + \frac{1}{2}f\mathsf{j} + c\mathsf{k}, \end{cases}$$

which may be written more conveniently for our purpose

$$(1') \qquad \begin{cases} \mathsf{a} = \dfrac{\partial \mathbf{D}}{\partial x} + \frac{1}{2}\,\mathsf{i} \times \operatorname{curl} \mathbf{D}, \\ \mathsf{b} = \dfrac{\partial \mathbf{D}}{\partial y} + \frac{1}{2}\,\mathsf{j} \times \operatorname{curl} \mathbf{D}, \\ \mathsf{c} = \dfrac{\partial \mathbf{D}}{\partial z} + \frac{1}{2}\,\mathsf{k} \times \operatorname{curl} \mathbf{D}. \end{cases}$$

Now any self-conjugate dyadic is expressible in the form*

$$\Omega' = a_0\mathsf{i}'\mathsf{i}' + b_0\mathsf{j}'\mathsf{j}' + c_0\mathsf{k}'\mathsf{k}',$$

where i', j', k' is another set of mutually perpendicular unit vectors. If the direction cosines of these two sets relative to each other are as in the table

	i'	j'	k'
i	l_1	m_1	n_1
j	l_2	m_2	n_2
k	l_3	m_3	n_3

* Gibbs-Wilson, *Vector Analysis*, p. 305.

then expressing Ω' in nonion form in terms of i, j, k, and identifying with Ω we obtain the known relations

$$(2)\qquad \begin{cases} a = a_0 l_1^2 + b_0 m_1^2 + c_0 n_1^2, \\ b = a_0 l_2^2 + b_0 m_2^2 + c_0 n_2^2, \\ c = a_0 l_3^2 + b_0 m_3^2 + c_0 n_3^2, \\ f = 2(a_0 l_2 l_3 + b_0 m_2 m_3 + c_0 n_2 n_3), \\ g = 2(a_0 l_3 l_1 + b_0 m_3 m_1 + c_0 n_3 n_1), \\ h = 2(a_0 l_1 l_2 + b_0 m_1 m_2 + c_0 n_1 n_2). \end{cases}$$

And because the scalar of the dyadic is an invariant*

$$(3)\qquad a + b + c = a_0 + b_0 + c_0 = \theta \text{ (say)}.$$

From the definition of Ω it also follows immediately that

$$(4)\qquad \begin{cases} \mathrm{a} = \mathrm{i} \cdot \Omega = \mathrm{i} \cdot \Omega', \\ \mathrm{b} = \mathrm{j} \cdot \Omega = \mathrm{j} \cdot \Omega', \\ \mathrm{c} = \mathrm{k} \cdot \Omega = \mathrm{k} \cdot \Omega', \end{cases}$$

in which the dyadic, being self-conjugate, may be written as a prefactor if desired.

Stress Relations.

The directions of i′, j′, k′ are the principal axes of strain,† and therefore also the principal axes of stress because the body is isotropic. Denote the principal stresses by P_1, P_2, P_3 and their tensors by p_1, p_2, p_3; and let P_x, P_y, P_z be the stresses across elementary planes normal to i, j, k respectively. Then considering the equilibrium of the elementary tetrahedron, three of whose edges are parallel to i′, j′, k′, while the face cutting these is perpendicular to i, we find the relations

$$\begin{cases} \mathrm{P}_x \cdot \mathrm{i} = p_1 l_1^2 + p_2 m_1^2 + p_3 n_1^2, \\ \mathrm{P}_x \cdot \mathrm{j} = p_1 l_1 l_2 + p_2 m_1 m_2 + p_3 n_1 n_2, \\ \mathrm{P}_x \cdot \mathrm{k} = p_1 l_1 l_3 + p_2 m_1 m_3 + p_3 n_1 n_3. \end{cases}$$

Hence the stress P_x has the value

$$\mathrm{P}_x = \mathrm{i} \cdot (\mathrm{P}_1 \mathrm{i}' + \mathrm{P}_2 \mathrm{j}' + \mathrm{P}_3 \mathrm{k}') = \mathrm{i} \cdot \Lambda,$$

* *Ibid.*, p. 319—320. † *Ibid.*, p. 351.

where Λ is the dyadic

$$\Lambda = \mathbf{P}_1\mathbf{i}' + \mathbf{P}_2\mathbf{j}' + \mathbf{P}_3\mathbf{k}',$$

which is self-conjugate because $\mathbf{P}_1 = p_1\mathbf{i}'$, &c. The above and two similar relations may be combined in the formulæ

$$(5) \qquad \begin{cases} \mathbf{P}_x = \mathbf{i} \cdot \Lambda, \\ \mathbf{P}_y = \mathbf{j} \cdot \Lambda, \\ \mathbf{P}_z = \mathbf{k} \cdot \Lambda. \end{cases}$$

from which we may construct the dyadic

$$(6) \qquad \mathbf{i}\mathbf{P}_x + \mathbf{j}\mathbf{P}_y + \mathbf{k}\mathbf{P}_z = \mathbf{I} \cdot \Lambda = \Lambda,$$

which is therefore self-conjugate, being identical with Λ.

The equations (5) give the stress across any elementary plane in terms of the principal stresses at that point. Hence it follows that at any surface σ, whether the surface of the body or another surface drawn in the body, the stress $\mathbf{P}_n$ at an element whose normal is $\mathbf{n}$ (a unit vector) has the value

$$(5') \qquad \mathbf{P}_n = \Lambda \cdot \mathbf{n}.$$

If the bodily force per unit mass be $\mathbf{F}$, it follows that for the equilibrium of the mass within any closed surface σ

$$\int \rho \mathbf{F} d\tau = -\int \mathbf{P}_n d\sigma = -\int \Lambda \cdot \mathbf{n} d\sigma,$$

$d\tau$ denoting an element of volume. Transforming the surface integral to a volume integral,* we have

$$\int \rho \mathbf{F} d\tau = \int \nabla \cdot \Lambda d\tau ;$$

and since this is true for any volume whatever in the body, it follows that for every point

$$\rho \mathbf{F} = \nabla \cdot \Lambda$$

$$= \frac{\partial}{\partial x}\mathbf{P}_x + \frac{\partial}{\partial y}\mathbf{P}_y + \frac{\partial}{\partial z}\mathbf{P}_z ;\dagger$$

or, since Λ is self-conjugate,

$$(7) \qquad \rho \mathbf{F} = \mathbf{i}\ \mathrm{div}\ \mathbf{P}_x + \mathbf{j}\ \mathrm{div}\ \mathbf{P}_y + \mathbf{k}\ \mathrm{div}\ \mathbf{P}_z.$$

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* *Ibid.*, p. 409. † *Ibid.*, p. 406.

This is the "indefinite" equation of equilibrium. The corresponding equation for vibrations is, by D'Alembert's principle,

$$(7') \qquad \rho\mathbf{F}-\rho\ddot{\mathbf{D}}=\mathbf{i}\,\mathrm{div}\,\mathbf{P}_x+\mathbf{j}\,\mathrm{div}\,\mathbf{P}_y+\mathbf{k}\,\mathrm{div}\,\mathbf{P}_z.$$

Stress-Strain Relations and the Definite Equation of Equilibrium.

To establish the definite equation of equilibrium we need to connect the stresses and the strains. This is most easily done in the case of the principal stresses. Consider an elementary parallelepiped whose edges are parallel to $\mathbf{i}', \mathbf{j}', \mathbf{k}'$, that is to the directions of the principal stresses. Due to the stress $\mathbf{P}_1$ the edges parallel to $\mathbf{i}'$ are reduced per unit length by p_1/E, where E is Young's modulus. Owing however to the stresses $\mathbf{P}_2$ and $\mathbf{P}_3$ the same edges are extended per unit length by $\sigma(p_2+p_3)/E$, σ denoting Poisson's ratio. But the coefficient of linear extension is represented by a_0. Hence

$$a_0=[\sigma(p_2+p_3)-p_1]/E.$$

Putting $p\equiv p_1+p_2+p_3$, and writing down the other two equations similar to the last, we have

$$(8) \qquad \begin{cases} a_0=[\sigma p-(1+\sigma)p_1]/E, \\ b_0=[\sigma p-(1+\sigma)p_2]/E, \\ c_0=[\sigma p-(1+\sigma)p_3]/E. \end{cases}$$

But we have seen that the sum of these is the invariant θ; hence

$$(8') \qquad \theta=-p(1-2\sigma)/E.$$

If now in the expression for Λ we substitute the values of $\mathbf{P}_1=\mathbf{i}'p_1$, $\mathbf{P}_2=\mathbf{j}'p_2$, $\mathbf{P}_3=\mathbf{k}'p_3$ given by (8) we find

$$\Lambda=-\frac{E}{1+\sigma}\left[\frac{\sigma\theta}{1-2\sigma}+a_0\right]\mathbf{i}'\mathbf{i}'-\ldots-\ldots,$$

or, using Lamé's constants,

$$(9) \qquad \Lambda=-\lambda\theta\mathbf{I}-2\mu\Omega.$$

This formula connects the stress dyadic Λ with the strain dyadic Ω.

From this relation the equations of equilibrium are obtained as follows: Substituting in (5) the value of Λ just obtained, we find

$$(10)\qquad \begin{cases} -\mathrm{P}_x = \mathrm{i}\lambda\theta + 2\mu\,\mathrm{a}, \\ -\mathrm{P}_y = \mathrm{j}\lambda\theta + 2\mu\,\mathrm{b}, \\ -\mathrm{P}_z = \mathrm{k}\lambda\theta + 2\mu\,\mathrm{c}. \end{cases}$$

On using these values of the stresses to transform the indefinite equation (7), and substituting the values of a, b, c given by (1′), we have at once

$$(11)\qquad \begin{cases} -\rho\mathrm{F} = \sum_{i,j,k} \mathrm{i}\ \mathrm{div}\left[\mathrm{i}\lambda\theta + \mu\left(2\dfrac{\partial \mathrm{D}}{\partial x} + \mathrm{i}\times\mathrm{curl}\,\mathrm{D}\right)\right] \\ \qquad = \lambda\ \mathrm{grad}\ \theta + 2\mu\ \mathrm{grad\ div}\ \mathrm{D} \\ \qquad\qquad - \mu\,\Sigma\mathrm{ii}\cdot\mathrm{curl\ curl}\ \mathrm{D} \\ \qquad = (\lambda+\mu)\ \mathrm{grad}\ \theta + \mu\nabla^2\mathrm{D}; \end{cases}$$

while the equation of vibrations becomes

$$(11')\qquad \rho(\ddot{\mathrm{D}} - \mathrm{F}) = \mu\nabla^2\mathrm{D} + (\lambda+\mu)\ \mathrm{grad}\ \theta.$$

Similarly the value of the *surface traction* T, in virtue of (5′), is given by

$$(12)\qquad \begin{aligned} -\mathrm{T} &= -\mathrm{n}\cdot\Lambda = \mathrm{n}\cdot[\lambda\theta\mathrm{I} + 2\mu\Omega] \\ &= \lambda\theta\mathrm{n} + 2\mu\,\mathrm{n}\cdot\left[\nabla\mathrm{D} + \tfrac{1}{2}\mathrm{I}\times\mathrm{curl}\,\mathrm{D}\right] \\ &= \lambda\theta\mathrm{n} + 2\mu\frac{d\mathrm{D}}{dn} + \mu\,\mathrm{n}\times\mathrm{curl}\,\mathrm{D}. \end{aligned}$$

The Case of Heredity.

When there is no residual elasticity the quantities λ and μ are two independent constants, and the equations (10) may be taken as the fundamental stress-strain relations. They give the stress across any elementary plane in terms of the cubical dilation and the strain vectors a, b, c. Now in the case of a body possessing heredity this stress depends not only on the actual values of θ and a, b, c, but on all the values they have previously assumed. Thus if heredity is linear, and the effect of the body's past history anterior to a certain instant is negligible, measuring the time variable from this

instant, we have, in place of the relations (10), others of the form

$$(13)\quad \begin{cases} -\mathbf{P}_x(t) = \mathrm{i}\lambda\theta(t) + 2\mu\,\mathrm{a}(t) + \int_0^t [\mathrm{i}\phi(t,\tau)\theta(\tau) + 2\psi(t,\tau)\,\mathrm{a}(\tau)]\,d\tau \\ -\mathbf{P}_y(t) = \mathrm{j}\lambda\theta(t) + 2\mu\,\mathrm{b}(t) + \int_0^t [\mathrm{j}\phi(t,\tau)\theta(\tau) + 2\psi(t,\tau)\,\mathrm{b}(\tau)]\,d\tau \\ -\mathbf{P}_z(t) = \mathrm{k}\lambda\theta(t) + 2\mu\,\mathrm{c}(t) + \int_0^t [\mathrm{k}\phi(t,\tau)\theta(\tau) + 2\psi(t,\tau)\mathrm{c}(\tau)]\,d\tau, \end{cases}$$

where $\phi(t, \tau)$ and $\psi(t, \tau)$ are two independent coefficients of heredity. If, following Volterra's notation, we introduce the operators A_1 and A_2 defined by

$$A_1 F(t) \equiv \mu F(t) + \int_0^t \psi(t, \tau)\, F(\tau)\, d\tau,$$

$$A_2 F(t) \equiv (\lambda + 2\mu)\, F(t) + \int_0^t [\phi(t, \tau) + 2\psi(t, \tau)]\, F(\tau)\, d\tau,$$

the preceding formulæ may be written

$$(13')\quad \begin{cases} -\mathbf{P}_x = \mathrm{i}\ (A_2 - 2A_1)\,\theta(t) + 2A_1\,\mathrm{a}(t), \\ -\mathbf{P}_y = \mathrm{j}\ (A_2 - 2A_1)\,\theta(t) + 2A_1\,\mathrm{b}(t), \\ -\mathbf{P}_z = \mathrm{k}\,(A_2 - 2A_1)\,\theta(t) + 2A_1\,\mathrm{c}(t). \end{cases}$$

Substituting these values in the indefinite equation (7), we find for the equilibrium of the body the *integro-differential equation*

$$(14)\quad \begin{cases} -\rho\mathbf{F} = \mu\nabla^2\mathbf{D} + (\lambda + \mu)\ \mathrm{grad}\ \theta \\ + \int_0^t [\psi(t, \tau)\nabla^2\mathbf{D}(t) + \{\phi(t, \tau) + \psi(t, \tau)\}\ \mathrm{grad}\ \theta(\tau)]\,d\tau, \end{cases}$$

which may also be written

$$(14')\qquad -\rho\mathbf{F} = A_1\nabla^2\mathbf{D} + (A_2 - A_1)\ \mathrm{grad}\ \theta,$$

all the quantities involved being functions of the time. The equation of vibrations is likewise

$$(15)\qquad \rho(\ddot{\mathbf{D}} - \mathbf{F}) = A_1\nabla^2\mathbf{D} + (A_2 - A_1)\ \mathrm{grad}\ \theta.$$

In finding the *surface traction* we now have in place of (9) the dyadic function of t

$$\Lambda(t) = -(A_2 - 2A_1)\,\theta(t)\,\mathbf{I} - 2A_1\,\Omega(t);$$

and therefore, in virtue of (5′), the surface traction $\mathbf{T}(t)$ is given by

$$(16)\quad \begin{cases} -\mathbf{T}(t) = -\mathbf{n}\cdot\Lambda(t) \\ = (A_2 - 2A_1)\,\mathbf{n}\theta(t) + 2A_1\dfrac{d\mathbf{D}(t)}{dn} + A_1\mathbf{n}\times\operatorname{curl}\mathbf{D}(t), \end{cases}$$

which, written in full, is

$$-\mathbf{T}(t) = \lambda\theta(t)\,\mathbf{n} + 2\mu\frac{d}{dn}\mathbf{D}(t) + \mu\,\mathbf{n}\times\operatorname{curl}\mathbf{D}(t)$$

$$+\int_0^t \left[\mathbf{n}\,\phi(t,\tau)\,\theta(\tau) + 2\psi(t,\tau)\frac{d}{dn}\mathbf{D}(\tau) + \psi(t,\tau)\,\mathbf{n}\times\operatorname{curl}\mathbf{D}(\tau)\right]d\tau.$$

A FORMULA IN ZONAL HARMONICS.

By *J. R. Wilton, M.A., D.Sc.*

THE following formula satisfied by the Legendre functions $P_n(x)$ and $Q_n(x)$ is of some importance, and is perhaps worth putting on record

$$(m-n)(m+n+1)\int_x^1 P_m(x)\,P_n(x)\,dx$$

$$= (1-x^2)(P_m'P_n - P_mP_n')$$

$$= \tfrac{1}{2}\{P_m[(n+1)P_{n+1} - nP_{n-1}] - P_n[(m+1)P_{m+1} - mP_{m-1}]\}.$$

The formula for $\int_x^1 P_n^2(x)\,dx$ is given in Whittaker's *Analysis*, Ex. 5, p. 236 (First Edition). The integral $\int Q_n^2(x)\,dx$ can be obtained in a similar form, but it involves $\int Q_0^2\,dx$, which depends upon $\int \log(1+x)\,\dfrac{dx}{1-x}$.

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SOME SIMPLE EXPANSION FORMULÆ.

By *G. N. Watson.*

1. A FORMULA for the quotient of two power series (supposed convergent) has been frequently given,* namely that, if $a_0 \neq 0$, then

$$\frac{\sum_{n=0}^{\infty} b_n x^n}{\sum_{n=0}^{\infty} a_n x^n} = \frac{1}{a_0} \sum_{n=0}^{\infty} c_n x^n,$$

where

$$c_n = \frac{(-)^n}{a_0^n} \begin{vmatrix} a_1 b_0 - a_0 b_1 & , a_0 & , 0 , 0 , \ldots, 0 \\ a_2 b_0 - a_0 b_2, & , a_1 & , a_0, 0 , \ldots, 0 \\ a_3 b_0 - a_0 b_3 & , a_2 & , a_1, a_0, \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1} b_0 - a_0 b_{n-1}, & a_{n-2}, & \ldots\ldots\ldots, & a_0 \\ a_n b_0 - a_0 b_n & , a_{n-1}, & \ldots\ldots\ldots, & a_1 \end{vmatrix}.$$

In this determinant the $(m+p+1)^{\text{th}}$ element of the m^{th} row vanishes, when $p = 1, 2, \ldots, n-m-1$.

In the special case in which the numerator is unity, this result gives

$$\left\{ \sum_{n=0}^{\infty} a_n x^n \right\}^{-1} = a_0^{-1} \sum_{n=0}^{\infty} g_{n,1} x^n,$$

where

$$g_{n,1} = \frac{(-)^n}{a_0^n} \begin{vmatrix} a_1 & , a_0 & , 0 , 0 , \ldots, 0 \\ a_2 & , a_1 & , a_0, 0 , \ldots, 0 \\ a_3 & , a_2 & , a_1, a_0, \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n-1}, & a_{n-2}, & \ldots\ldots\ldots, & a_0 \\ a_n & , a_{n-1}, & \ldots\ldots\ldots, & a_1 \end{vmatrix}.$$

A slightly different form of the result has been given by Mangeot,† namely,

* See, *e.g.*, Scott and Mathews, *Theory of Determinants* (1904), pp. 161–162, and Burnside and Panton, *Theory of Equations*, II., p. 63; I., p. 179.

† *Ann. de l'Ecole normale supérieure* (3), XIV., pp. 247–250.

$$g_{n,1}=\frac{(-)^n}{n!\,a_0^{\,n}}\begin{vmatrix} 2a_1, & a_0, & 0 & , & 0 & , \ldots, & 0 \\ 4a_2, & 3a_1, & 2a_0, & 0 & , \ldots, & 0 \\ 6a_3, & 5a_2, & 4a_1, & 3a_0, & \ldots, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ (2n-2)a_{n-1}, & \ldots, & (n-1)a_0 & & & \\ na_n, & (n-1)a_{n-1}, & \ldots, & a_1 & & \end{vmatrix},$$

but there does not seem to be any very simple direct method of transforming the two expressions for $g_{n,1}$ into each other. As the last row in Mangeot's determinant is not formed according to the same rules as the other rows, I shall call it the unnatural expression for $g_{n,1}$.

Mangeot has also given the corresponding formula for $\{\Sigma a_n x^n\}^{\frac{1}{2}}$, but, so far as I can find, it does not seem to have been noticed that both the natural and the unnatural expressions for $g_{n,1}$ are particular cases of some rather elegant formulæ for the coefficient of x^n in $\{\Sigma a_n x^n\}^{-\mu}$, where μ is unrestricted. It is, of course, well known that $\{\Sigma a_n x^n\}^{-\mu}$ can be expanded into a power series if $a_0 \neq 0$ and $|x|$ is sufficiently small. The formulæ are obtained by very obvious methods, and it is quite likely that they have been given before, though I have not been able to discover them elsewhere.*

§ 2. To obtain the required expansions write

$$\sum_{n=0}^{\infty} a_n x^n \equiv f(x),$$

$$\{f(x)\}^{-\mu} = a_0^{-\mu} \sum_{n=0}^{\infty} g_n x^n,$$

$$\{f(x)\}^{-\mu-1} = a_0^{-\mu-1} \sum_{n=0}^{\infty} h_n x^n.$$

Since

$$\frac{d}{dx}\log\{a_0^{-\mu}\sum_{n=0}^{\infty} g_n x^n\} = -\mu\frac{d}{dx}\log\{\sum_{n=0}^{\infty} a_n x^n\},$$

we have

$$\{\sum_{n=0}^{\infty} a_n x^n\}\{\sum_{n=1}^{\infty} n g_n x^{n-1}\} = -\mu\{\sum_{n=1}^{\infty} n a_n x^{n-1}\}\{\sum_{n=0}^{\infty} g_n x^n\}.$$

* Mangeot has given a formula for the coefficient of x^n in the product of a set of powers of power series, but this formula is a complicated one, expressing each coefficient linearly in terms of the preceding ones, the coefficients in the relation being determinants.

Equating coefficients of x^{n-1} on each side, we get

$$ng_n a_0 + (n-1)g_{n-1}a_1 + \ldots + 1.g_1 a_{n-1} = -\mu\left[g_{n-1}.1a_1 + g_{n-2}.2a_2 + \ldots + g_0.na_n\right],$$

that is to say

$$g_0\mu n a_n + g_1\{\mu(n-1)+1\}a_{n-1} + g_2\{\mu(n-2)+2\}a_{n-2} + \ldots + g_{n-1}(\mu.1 + n - 1)a_1 + g_n.na_0 = 0.$$

Eliminating $g_1, g_2, \ldots, g_{n-1}$ from the system of equations

$$g_0\mu a_1 + g_1.1a_0 = 0,$$

$$g_0\mu.2a_2 + g_1(\mu+1)a_1 + g_2.2a_0 = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\{g_0\mu n a_n + g_n n a_0\} + g_1\{\mu(n-1)+1\}a_1 + \ldots + g_{n-1}(\mu.1 + n - 1)a_1 = 0,$$

and remembering that $g_0 = 1$, we have at once

$$g_n = \frac{(-)^n}{n!\,a_0^n}\begin{vmatrix} \mu a_1, & 1.a_0, & 0, & 0, & \ldots, & 0 \\ 2\mu a_2, & (\mu+1)a_1, & 2a_0, & 0, & \ldots, & 0 \\ 3\mu a_3, & (2\mu+1)a_2, & (\mu+2)a_1, & 3a_0, & \ldots, & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ n\mu a_n, & \{(n-1)\mu+1\}a_{n-1}, & \ldots, & (\mu+n-1)a_1 & & \end{vmatrix}.$$

In this formula the coefficient of any of the numbers a_n is the coefficient of the preceding element in the same row diminished by $\mu - 1$. This is the natural formula for g_n.

To obtain the unnatural formula for g_n, we observe that

$$\frac{d}{dx}\{a_0^{-\mu}\sum_{n=0}^{\infty} g_n x^n\} = -\mu f'(x)\{a_0^{-\mu-1}\sum_{n=0}^{\infty} h_n x^n\},$$

that is to say

$$a_0\sum_{n=1}^{\infty} n g_n x^{n-1} = -\mu\{\sum_{n=1}^{\infty} n a_n x^{n-1}\}\{\sum_{n=0}^{\infty} h_n x^n\}.$$

Equating coefficients of x^{n-1} we get

$$na_0 g_n + \mu\{n a_n h_0 + (n-1)a_{n-1}h_1 + \ldots + 1.a_1 h_{n-1}\} = 0.$$

Also, from the obvious identity

$$a_0\sum_{n=0}^{\infty} g_n x^n \equiv \sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} h_n x^n,$$

we have, on equating coefficients of x^n,

$$a_0 g_n = a_n h_0 + a_{n-1}h_1 + \ldots + a_0 h_n.$$

Multiplying this by n and subtracting from the previous result, we have

$$(n\mu+n)\,a_n h_0+\{(n-1)\,\mu+n\}\,a_{n-1}h_1 + \{(n-2)\,\mu+n\}\,a_{n-2}h_2+\ldots+na_0h_n=0.$$

Eliminating $h_1, h_2, \ldots, h_{n-1}$ from the system of equations

$$(\mu+1)\,a_1h_0+1.a_0h_1 \qquad =0,$$

$$(2\mu+2)\,a_2h_0+(\mu+2)\,a_1h_1+2a_0h_2=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\{(n-1)\,\mu+(n-1)\}\,a_{n-1}h_0+\{(n-2)\,\mu+n-1\}\,a_{n-2}h_1+\ldots + (n-1)\,a_0h_{n-1}=0,$$

$$\{na_0g_n+n\mu a_nh_0\}+(n-1)\,\mu a_{n-1}h_1+\ldots+\mu a_1h_{n-1}=0,$$

and remembering that $h_0=1$, we have at once

$$g_n=\frac{(-)^n}{n!\,a_0^{\,n}}\begin{vmatrix} (\mu+1)\ a_1, & 1.a_0 & , 0 & , 0 & , \ldots, 0 \\ (2\mu+2)\,a_2, & (\mu+2)\,a_1 & , 2a_0 & , 0 & , \ldots, 0 \\ (3\mu+3)\,a_3, & (2\mu+3)\,a_2, & (\mu+3)\,a_1, & 3a_0, & \ldots, 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \{(n-1)\mu+n-1\}a_{n-1}, & \{(n-2)\mu+n-1\}a_{n-2}, & \ldots, & (n-1)a_0 & \\ n\mu a_n, & (n-1)\,\mu a_{n-1}, & \ldots\ldots\ldots\ldots, & \mu a_1 & \end{vmatrix}.$$

In this formula the coefficient of any of the numbers a_n is the coefficient of the preceding element in the same row diminished by μ. This is the unnatural formula for g_n.

3. If the expansion of $\exp\{\sum_{n=1}^{\infty} a_nx^n\}$ is $\sum_{n=0}^{\infty} d_nx^n$, from the identity

$$\sum_{n=1}^{\infty} nd_nx^{n-1}\equiv\frac{d}{dx}\exp\{\sum_{n=1}^{\infty} a_nx^n\} \equiv\{\sum_{n=1}^{\infty} na_nx^{n-1}\}\{\sum_{n=0}^{\infty} d_nx^n\},$$

we obtain the formula

$$nd_n=a_1d_{n-1}+2a_2d_{n-2}+\ldots+(n-1)\,a_{n-1}d_1+na_nd_0,$$

and hence, since $d_0 = 1$, we have

$$d_n = \frac{1}{n!} \begin{vmatrix} a_1, & -1, & 0, & 0, & \ldots\ldots\ldots\ldots, & 0 \\ 2a_2, & a_1, & -2, & 0, & \ldots\ldots\ldots\ldots, & 0 \\ 3a_3, & 2a_2, & a_1, & -3, & \ldots\ldots\ldots\ldots, & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots \\ (n-1)a_{n-1}, & (n-2)a_{n-2}, & \ldots, & a_1, & -(n-1) \\ na_n, & (n-1)a_{n-1}, & \ldots, & 2a_2, & a_1 \end{vmatrix},$$

This result, also, I have not encountered elsewhere.

ORDERS OF OPERATORS OF CONGRUENCE GROUPS MODULO $2^r 3^s$.

By *G. A. Miller.*

It is well known that the group formed by the $\phi(m)$ positive integers which do not exceed the positive integer m and are prime to m, when these numbers are combined by multiplication mod m, is simply isomorphic with the group of isomorphisms of the cyclic group of order m. Moreover, every possible automorphism of an Abelian group may be obtained by establishing an $(\alpha, 1)$ isomorphism between this Abelian group and itself or one of its sub-groups, and then letting each operator of the group correspond to the product obtained by multiplying it into the corresponding operator in this isomorphism.

On the other hand, if we establish any $(\alpha, 1)$ isomorphism between an Abelian group G and itself or one of its sub-groups and then multiply the corresponding operators, these products constitute a group whose operators may be arranged in a (1, 1) correspondence with their constituents from G whenever none of these constituents except the identity corresponds to its inverse in the said $(\alpha, 1)$ isomorphism. When other constituents than the identity correspond to their inverses in this $(\alpha, 1)$ isomorphism all such constituents form a sub-group of G and the sub-group obtained by multiplying the corresponding operators in the said isomorphism is simply isomorphic with the quotient group of G with respect to the sub-group formed by the operators corresponding to their inverses.

These considerations enable us to unify the problems of

finding the group of automorphisms of an Abelian group and establishing $(\alpha, 1)$ correspondences between this group and itself or its sub-groups.* It is only necessary to consider all such possible correspondences in which no operator besides the identity corresponds to its inverse. The orders of these correspondences are not always evident. When m is a power of two these orders can, however, be very easily determined in the following manner:

Let G be the cyclic group of order 2^β and consider any automorphism of G corresponding to an isomorphism in which α is divisible by 4. In the square of such an automorphism α is clearly divisible by exactly one higher power of 2 than in the automorphism itself. Hence the order of such an automorphism is equal to the order of the sub-group of G which is made isomorphic with G to establish the automorphism. That is, if a number is congruent to 1 mod 2^γ, $\gamma > 1$, but not congruent to 1 mod $2^{\gamma+1}$, the exponent to which it belongs mod 2^β, $\beta > 3$, is $2^{\beta-\gamma}$.

It remains only to consider the numbers which are concongruent to 3 or -1 mod 4. It is clear that -1 mod 2^β corresponds to exponent 2, and that the other numbers in question are the products of this number and the numbers considered in the preceding paragraph. Since each of the latter numbers, besides the identity, belongs to an even exponent, these products belong to the same exponent as the factor from the latter numbers whenever this factor is not unity. Hence there results the theorem: *If a number is* $\equiv \pm 1$ *mod* 2^γ *but not* $\equiv \pm 1$ *mod* $2^{\gamma+1}$, $1 < \gamma < \beta$, *the exponent to which it belongs mod* 2^β *is* $2^{\beta-\gamma}$.

This theorem enables us to determine directly the exponent to which every odd number, except those $\equiv \pm 1$ mod 2^β, belongs mod 2^β, and the exponents to which these two exceptional numbers belong are known to be one and two respectively. As the order of each of the operators of the congruence groups formed by the $\phi(2^\beta)$ positive integers which do not exceed 2^β, when these numbers are combined by multiplication mod 2^β, is equal to the exponent to which the corresponding number belongs mod 2^β, the above theorem enables us to state directly the order of each of these operators.

When m is a power of an odd prime number p^β the orders of those among the $\phi(m)$ positive integers which do not exceed m and are both prime to m and congruent to 1

* G. A. Miller, *Bulletin of the American Mathematical Society*, vol. vi. (1900), p. 337; Miller, Blichfeldt, Dickson, *Finite Groups* (1916), p. 103.

mod p can be directly obtained from the fact that *whenever a cyclic group of order* p^β, p *being an odd prime number, is made isomorphic with one of its proper sub-groups and each operator is made to correspond to the product obtained by multiplying it into its corresponding operator in this isomorphism, the order of the automorphism thus obtained is equal to the order of the said proper sub-group.*

In other words, the exponent to which a number $\equiv 1$ mod p belongs mod p^β is $p^{\beta-\gamma}$, whenever this number is $\equiv 1$ mod p^γ, $\beta > \gamma \geq 1$, but $\not\equiv 1$ mod $p^{\gamma+1}$, and only then. The negatives of each of these numbers clearly belongs to the double of the exponent of the corresponding positive one, since -1 belongs to exponent 2. Hence the exponent to which any number $\equiv \pm 1$ mod p belongs mod p^β can be directly stated. In particular, the order of each operator formed by the $\phi(3^\beta)$ numbers which are prime to 3 and less than 3^β can be directly found according to the given theorems.

Since the group of isomorphisms of any Abelian group is the direct product of the groups of isomorphisms of its Sylow sub-groups the above considerations enable us to find directly the order of each operator in any congruence group mod $2^r 3^s$ where r and s are positive integers or zero, formed by the $\phi(2^r 3^s)$ numbers which do not exceed $2^r 3^s$ and are prime to this number. In fact, it is only necessary to find the order of this operator with respect to the two moduli 2^r and 3^s, and then find the least common multiple of these orders. For instance, to find the order of the operator represented by the number 7 mod 36 it may be noted that 7 belongs to exponent 2 mod 4 and to exponent 3 mod 9 according to the given general theorems. Hence the order of the operator corresponding to 7 mod 36 is 6. In particular, the above method enables us to see directly that the primitive roots of any power of 3 are the positive integers which are congruent to -1 mod 3 but not congruent to -1 mod 9.

To exhibit direct contact of this note it may be observed that in Weber's *Lehrbuch des Algebra*, second edition, vol. ii. (1899), p. 65, it is proved by special considerations that the number 5 belongs to exponent $2^{\beta-2}$ mod 2^β, and in a footnote it is observed that 3 can be proved similarly to belong to this exponent. A somewhat similar method of procedure is employed by P. Bachmann in his *Niedere Zahlentheorie*, Erster Teil (1902), p. 331. On the other hand, the general method explained above avoids such special considerations and exhibits this result as a very special consequence of establishing automorphisms of Abelian groups.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLIV.

On certain multiple integrals and series which occur in the analytic theory of numbers.

1. In the course of some recent researches* in the theory of numbers I have had to consider certain multiple series of the type

$$(1) \qquad \Sigma m_1^{\alpha_1-1} m_2^{\alpha_2-1} \dots m_{p+q}^{\alpha_{p+q}-1} f\{|M_{1,p}-M_{p+1,p+q}|\},$$

where

$$M_{1,p}=a_1m_1^c+a_2m_2^c+\dots+a_pm_p^c,$$

$$M_{p+1,p+q}=a_{p+1}m_{p+1}^c+a_{p+2}m_{p+2}^c+\dots+a_{p+q}m_{p+q}^c,$$

c and the a's being positive, and where the field of summation comprises all positive values of the m's, or all that do not satisfy some special condition. As typical examples I may mention the series

$$\sum_{m\neq n} \frac{(mn)^{\alpha-1}}{|\sqrt{m}-\sqrt{n}|}, \quad \sum_{m\neq n} (mn)^{\alpha-1} \left| \frac{\sin 2\pi\omega\,(\sqrt{m}-\sqrt{n})}{\sqrt{m}-\sqrt{n}} \right|,$$

and

$$\Sigma \frac{(mnpq)^{\alpha-1}}{|\sqrt{m}+\sqrt{n}-\sqrt{p}-\sqrt{q}|},$$

where the summation is extended to all positive integral values of m, n, p, and q such that

$$\sqrt{m}+\sqrt{n}\neq\sqrt{p}+\sqrt{q}.$$

The series are interesting in themselves, and it seems worth while to investigate more general criteria for their convergence.

It is, however, natural to consider first the corresponding integrals of the type

$$(2) \qquad \int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \dots x_{p+q}^{\alpha_{p+q}-1} f\{|X_{1,p}-X_{p+1,p+q}|\}\,dx_1dx_2\dots dx_{p+q},$$

* See G. H. Hardy, 'The average order of the arithmetical functions $P(x)$ and $\Delta(x)$', *Proc. London Math. Soc.*, ser. 2, vol xv. (1916), pp. 192–213.

where $$X_{1,p} = a_1 x_1^{\,c} + a_2 x_2^{\,c} + \ldots + a_p x_p^{\,c},$$

$$X_{p+1,\,p+q} = a_{p+1} x_{p+1}^{c} + a_{p+2} x_{p+2}^{c} + \ldots + a_{p+q} x_{p+q}^{c},$$

and the field of integration comprises all positive values of the x's.

It is evident that the integral is not, in general, convergent unless all the α's are positive. I shall suppose that this condition is satisfied, and that f is summable and never negative.

2. Theorem. *The integral* (2) *is convergent if*

$$\alpha_1 > 0,\ \alpha_2 > 0,\ \ldots,\ \alpha_{p+q} > 0,$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_{p+q} < c,$$

and the integral

$$\int_0^\infty t^{(\alpha_1+\alpha_2+\ldots+\alpha_{p+q})/c-1} f(t)\, dt \tag{3}$$

is convergent. Its value is then equal to

$$\frac{1}{\pi}\, c^{-p-q} \prod_1^{p+q} \left\{ a_\nu^{-\alpha_\nu/c}\, \Gamma\left(\frac{\alpha_\nu}{c}\right)\right\} \Gamma\left(1 - \frac{1}{c}\sum_1^{p+q}\alpha_\nu\right)\left\{\sin\left(\frac{\pi}{c}\sum_1^{p}\alpha_\nu\right) + \sin\left(\frac{\pi}{c}\sum_{p+1}^{p+q}\alpha_\nu\right)\right\}$$

times the value of the integral (3).

In proving this theorem we may suppose, without real loss of generality, that

$$a_1 = a_2 = \ldots = a_{p+q} = 1, \quad c = 1;$$

for the general case is reduced to this particular one when we effect the substitution

$$a_1 x_1^{\,c} = y_1,\ a_2 x_2^{\,c} = y_2,\ \ldots,\ a_{p+q} x_{p+q}^{c} = y_{p+q}.$$

By the well-known formula of Dirichlet and Liouville, we have

$$\int x_1^{\alpha_1-1} x_2^{\alpha_2-1} \ldots x_p^{\alpha_p-1} f\left|\, X_{1,p} - X_{p+1,\,p+q} \,\right| dx_1\, dx_2 \ldots dx_p$$

$$= \frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_p)}{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_p)} \int_0^\infty f\left|\, \xi - X_{p+1,\,p+q} \,\right| \xi^{\alpha_1+\alpha_2+\ldots+\alpha_p-1}\, d\xi,$$

provided that the last integral is convergent. Hence the integral (2) is equal to

$$A\int x_{p+1}^{\alpha_{p+1}-1}\dots x_{p+q}^{\alpha_{p+q}-1}dx_{p+1}\dots dx_{p+q}\int_0^\infty f|\xi-X_{p+1,\,p+q}|\,\xi^{\alpha_1+\dots+\alpha_p-1}\,d\xi$$

$$=A\int_0^\infty \xi^{\alpha_1+\dots+\alpha_p-1}\,d\xi\int x_{p+1}^{\alpha_{p+1}-1}\dots$$

$$\times x_{p+q}^{\alpha_{p+q}-1}f|\xi-X_{p+1,\,p+q}|\,dx_{p+1}\dots dx_{p+q}$$

$$=AB\int_0^\infty \xi^{\alpha_1+\dots+\alpha_p-1}d\xi\int_0^\infty \eta^{\alpha_{p+1}+\dots+\alpha_{p+q}-1}f|\xi-\eta|\,d\eta$$

$$=AB\int \xi^{\lambda-1}\eta^{\mu-1}f|\xi-\eta|\,d\eta,$$

where $\lambda=\alpha_1+\alpha_2+\dots+\alpha_p$, $\mu=\alpha_{p+1}+\alpha_{p+2}+\dots+\alpha_{p+q}$,

$$A=\frac{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\dots\Gamma(\alpha_p)}{\Gamma(\lambda)},\quad B=\frac{\Gamma(\alpha_{p+1})\,\Gamma(\alpha_{p+2})\dots\Gamma(\alpha_{p+q})}{\Gamma(\mu)};$$

provided that the double integral in ξ and η is convergent.

In order to reduce the double integral further, we write

$$J=\int\xi^{\lambda-1}\eta^{\mu-1}f|\xi-\eta|\,d\xi\,d\eta=K+L,$$

where K and L are double integrals extended respectively over the fields for which $\xi>\eta$ and $\xi<\eta$. We have

$$K=\int_0^\infty \eta^{\mu-1}\,d\eta\int_\eta^\infty \xi^{\lambda-1}f(\xi-\eta)\,d\xi$$

$$=\int_0^\infty \eta^{\mu-1}\,d\eta\int_0^\infty (\eta+\zeta)^{\lambda-1}f(\zeta)\,d\zeta$$

$$=\int_0^\infty f(\zeta)\,d\zeta\int_0^\infty (\eta+\zeta)^{\lambda-1}\eta^{\mu-1}\,d\eta$$

$$=\frac{\Gamma(\mu)\,\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)}\int_0^\infty \zeta^{\lambda+\mu-1}f(\zeta)\,d\zeta,$$

provided that $\mu>0$, $\lambda+\mu<1$, and the integral last written is convergent. Similarly

$$L=\frac{\Gamma(\lambda)\,\Gamma(1-\lambda-\mu)}{\Gamma(1-\mu)}\int_0^\infty \zeta^{\lambda+\mu-1}f(\zeta)\,d\zeta,$$

if $\lambda > 0$, $\lambda + \mu < 1$. We have therefore

$$J = \left\{ \frac{\Gamma(\lambda)}{\Gamma(1-\mu)} + \frac{\Gamma(\mu)}{\Gamma(1-\lambda)} \right\} \Gamma(1-\lambda-\mu) \int_0^\infty \zeta^{\lambda+\mu-1} f(\zeta)\, d\zeta,$$

if $\lambda > 0$, $\mu > 0$, $\lambda + \mu < 1$, and the integral is convergent.

If now we observe that

$$\frac{1}{\Gamma(\lambda)\Gamma(\mu)} \left\{ \frac{\Gamma(\lambda)}{\Gamma(1-\mu)} + \frac{\Gamma(\mu)}{\Gamma(1-\lambda)} \right\} = \frac{1}{\pi} (\sin \lambda\pi + \sin \mu\pi),$$

we find that

$$\int x_1^{\alpha_1 - 1} \dots x_{p+q}^{\alpha_{p+q}-1} f\left| x_1 + \dots + x_p - x_{p+1} - \dots - x_{p+q} \right| dx_1 \dots dx_{p+q}$$

$$= \frac{1}{\pi} \prod_1^{p+q} \{\Gamma(\alpha_\nu)\} \Gamma\left(1 - \sum_1^{p+q} \alpha_\nu\right) \left\{ \sin\left(\pi \sum_1^{p} \alpha_\nu\right) + \sin\left(\pi \sum_{p+1}^{p+q} \alpha_\nu\right) \right\} \int_0^\infty \zeta^{\Sigma\alpha - 1} f(\zeta)\, d\zeta,$$

provided only that

$$\alpha_1 > 0,\ \alpha_2 > 0,\ \dots,\ \alpha_{p+q} > 0,\ \sum_1^{p+q} \alpha_\nu < 1,$$

and that the integral with respect to ζ is convergent. This result is, as I have already explained, equivalent to that stated in the theorem.

3. Suppose in particular that

$$\alpha_1 = \alpha_2 = \dots = \alpha_{p+q} = \alpha,$$
$$c = 1,$$
$$f(\zeta) = \left| \frac{\sin 2\pi\omega\zeta}{\zeta} \right|.$$

Then the integral

$$\int (x_1 x_2 \dots x_{p+q})^{\alpha-1} \left| \frac{\sin 2\pi\omega\,(x_1 + \dots + x_p - x_{p+1} - \dots - x_{p+q})}{x_1 + \dots + x_p - x_{p+1} - \dots - x_{p+q}} \right| dx_1 \dots dx_{p+q}$$

is convergent if

$$0 < \alpha < \frac{1}{p+q}.$$

If c is not equal to 1, the condition is

$$0 < \alpha < \frac{c}{p+q}.$$

This is of course not really a distinct result. I mention it explicitly in order to compare it later with the results which I shall obtain in another note concerning the corresponding multiple series.

NOTE ON A DETERMINANT WHOSE ELEMENTS ARE ALEPH FUNCTIONS.

By *Sir Thomas Muir, LL.D.*

1. In the *Messenger of Mathematics*, vol. xxi. (1892), pp. 148—157, there is given an important theorem which does not seem to have received the attention due to it. It concerns a determinant of the kind here specified, and in effect is that

$$\frac{\begin{vmatrix} (\,)^{\alpha+\alpha'} & (\,)^{\alpha+\beta'} & (\,)^{\alpha+\gamma'} \dots \\ (\,)^{\beta+\alpha'} & (\,)^{\beta+\beta'} & (\,)^{\beta+\gamma'} \dots \\ (\,)^{\gamma+\alpha'} & (\,)^{\gamma+\beta'} & (\,)^{\gamma+\gamma'} \dots \\ \dots\dots & \dots\dots & \dots\dots \end{vmatrix}}{(abc\dots)^{n-1}} = (-1)^{\frac{1}{2}n(n-1)} \frac{\begin{vmatrix} S_{\alpha+\alpha'} & S_{\alpha+\beta'} & S_{\alpha+\gamma'} \dots \\ S_{\beta+\alpha'} & S_{\beta+\beta'} & S_{\beta+\gamma'} \dots \\ S_{\gamma+\alpha'} & S_{\gamma+\beta'} & S_{\gamma+\gamma'} \dots \\ \dots\dots & \dots\dots & \dots\dots \end{vmatrix}}{\zeta(a, b, c, \dots)},$$

where

$$(\,)^{\theta} \equiv (a, b, c, \dots)^{\theta} \equiv \theta^{\text{th}} \text{ Aleph function of } a, b, c, \dots,$$

$$S_{\theta} \equiv a^{\theta} + b^{\theta} + c^{\theta} + \dots,$$

and the number of the variables $a, b, c, \dots$ is n. The original mode of proof suffers from being indirect, and from its steps being seemingly inconsequent.

2. Since the determinant on the right-hand is equal to the product of

$$|a^{\alpha} b^{\beta} c^{\gamma} \dots| \text{ and } |a^{\alpha'} b^{\beta'} c^{\gamma'} \dots|,$$

and therefore contains $\zeta(a, b, c, \dots)$ as a factor, it follows that the determinant on the left must have $(abc\dots)^{n-1}$ as a factor, and that the cofactor in the latter case must be the same as in the former case, namely,

$$\begin{vmatrix} (\,)^{\alpha} & (\,)^{\beta} & (\,)^{\gamma} & \dots \\ (\,)^{\alpha-1} & (\,)^{\beta-1} & (\,)^{\gamma-1} & \dots \\ (\,)^{\alpha-2} & (\,)^{\beta-2} & (\,)^{\gamma-2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix} \cdot \begin{vmatrix} (\,)^{\alpha'} & (\,)^{\beta'} & (\,)^{\gamma'} & \dots \\ (\,)^{\alpha'-1} & (\,)^{\beta'-1} & (\,)^{\gamma'-1} & \dots \\ (\,)^{\alpha'-2} & (\,)^{\beta'-2} & (\,)^{\gamma'-2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix}$$

or differ only in sign.

3. The author of the theorem, Mr. H. W. Segar, in speaking of it as an extension of Jacobi's theorem regarding the quotient of $|a^{\alpha} b^{\beta} c^{\gamma} \dots|$ by $\zeta^{\frac{1}{2}}(a, b, c, \dots)$ is hardly accurate enough and is certainly a little unfair to himself. Even the

mode in which it is shown to include Jacobi's is evidence of this, and the direct mode of proof herewith given strengthens the view.

4. Using the fact that every Aleph function is expressible as the quotient of an alternant by the difference-product of the variables, namely,

$$(a,\ b,\ c)^\theta = \frac{|a^0\, b^1\, c^{2+\theta}|}{|a^0\, b^1\, c^2|},$$

and confining ourselves to *three* variables, we see that the determinant on the left of the asserted equality

$$= \begin{vmatrix} |a^0 b^1 c^{2+\alpha+\alpha'}| & |a^0 b^1 c^{2+\alpha+\beta'}| & |a^0 b^1 c^{2+\alpha+\gamma'}| \\ |a^0 b^1 c^{2+\beta+\alpha'}| & |a^0 b^1 c^{2+\beta+\beta'}| & |a^0 b^1 c^{2+\beta+\gamma'}| \\ |a^0 b^1 c^{2+\gamma+\alpha'}| & |a^0 b^1 c^{2+\gamma+\beta'}| & |a^0 b^1 c^{2+\gamma+\gamma'}| \end{vmatrix} \div \zeta^3,$$

and therefore by the multiplication-theorem

$$= \begin{vmatrix} a^\alpha(c-b) & b^\alpha(a-c) & c^\alpha(b-a) \\ a^\beta(c-b) & b^\beta(a-c) & c^\beta(b-a) \\ a^\gamma(c-b) & b^\gamma(a-c) & c^\gamma(b-a) \end{vmatrix} \cdot \begin{vmatrix} a^{2+\alpha'} & b^{2+\alpha'} & c^{2+\alpha'} \\ a^{2+\beta'} & b^{2+\beta'} & c^{2+\beta'} \\ a^{2+\gamma'} & b^{2+\gamma'} & c^{2+\gamma'} \end{vmatrix} \div \zeta^3,$$

$$= (c-b)(a-c)(b-a)\,|a^\alpha b^\beta c^\gamma| \,.\, a^2 b^2 c^2\, |a^{\alpha'} b^{\beta'} c^{\gamma'}| \div \zeta^3,$$

$$= -\, a^2 b^2 c^2 . \frac{|a^\alpha b^\beta c^\gamma| \,.\, |a^{\alpha'} b^{\beta'} c^{\gamma'}|}{\zeta};$$

and therefore the entire left-hand member

$$= -\begin{vmatrix} S_{\alpha+\alpha'} & S_{\alpha+\beta'} & S_{\alpha+\gamma'} \\ S_{\beta+\alpha'} & S_{\beta+\beta'} & S_{\beta+\gamma'} \\ S_{\gamma+\alpha'} & S_{\gamma+\beta'} & S_{\gamma+\gamma'} \end{vmatrix} \div \zeta,$$

as required.

(5) Taking the case where $\alpha', \beta', \gamma', \ldots = 0, 1, 2, \ldots$ we obtain

$$\frac{\begin{vmatrix} ()^\alpha & ()^{\alpha+1} & ()^{\alpha+2} & \ldots \\ ()^\beta & ()^{\beta+1} & ()^{\beta+2} & \ldots \\ ()^\gamma & ()^{\gamma+1} & ()^{\gamma+2} & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \end{vmatrix}}{(abc\ldots)^{n-1}} = (-1)^{\frac{1}{2}n(n-1)} \begin{vmatrix} ()^\alpha & ()^\beta & ()^\gamma & \ldots \\ ()^{\alpha-1} & ()^{\beta-1} & ()^{\gamma-1} & \ldots \\ ()^{\alpha-2} & ()^{\beta-2} & ()^{\gamma-2} & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \end{vmatrix}.$$

This is readily proved independently by using Jacobi's equivalent for each of the determinants involved; and, having so obtained it, we may substitute its left-hand member for the right-hand member in Segar's general theorem, and thereby evolve the simple-looking alternative form

$$\begin{vmatrix} (\,)^{\alpha+\alpha'} & (\,)^{\alpha+\beta'} & (\,)^{\alpha+\gamma'} & \dots \\ (\,)^{\beta+\alpha'} & (\,)^{\beta+\beta'} & (\,)^{\beta+\gamma'} & \dots \\ (\,)^{\gamma+\alpha'} & (\,)^{\gamma+\beta'} & (\,)^{\gamma+\gamma'} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix}$$

$$= \begin{vmatrix} (\,)^{\alpha} & (\,)^{\beta} & (\,)^{\gamma} & \dots \\ (\,)^{\alpha+1} & (\,)^{\beta+1} & (\,)^{\gamma+1} & \dots \\ (\,)^{\alpha+2} & (\,)^{\beta+2} & (\,)^{\gamma+2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix} \cdot \begin{vmatrix} (\,)^{\alpha'} & (\,)^{\beta'} & (\,)^{\gamma'} & \dots \\ (\,)^{\alpha'-1} & (\,)^{\beta'-1} & (\,)^{\gamma'-1} & \dots \\ (\,)^{\alpha'-2} & (\,)^{\beta'-2} & (\,)^{\gamma'-2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix}.$$

Further, the left-hand member here being substantially unaltered by the interchange of $\alpha, \beta, \gamma, \dots$, and $\alpha', \beta', \gamma', \dots$, the right-hand member is seen to be equal to

$$\begin{vmatrix} (\,)^{\alpha} & (\,)^{\beta} & (\,)^{\gamma} & \dots \\ (\,)^{\alpha-1}(\,)^{\beta-1}(\,)^{\gamma-1} & \dots \\ (\,)^{\alpha-2}(\,)^{\beta-2}(\,)^{\gamma-2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix} \cdot \begin{vmatrix} (\,)^{\alpha'} & (\,)^{\beta'} & (\,)^{\gamma'} & \dots \\ (\,)^{\alpha'+1}(\,)^{\beta'+1}(\,)^{\gamma'+1} & \dots \\ (\,)^{\alpha'+2}(\,)^{\beta'+2}(\,)^{\gamma'+2} & \dots \\ \dots\dots & \dots\dots & \dots\dots & \end{vmatrix}.$$

The special case where $\alpha', \beta', \gamma', \dots = \alpha, \beta, \gamma, \dots$ will also be found of interest.*

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* I take the opportunity of noting that the extension which Prof. Burnside gives (*Messenger of Mathematics*, xlv., pp. 183–185) of one of the theorems of my paper (*Messenger of Mathematics*, xlv., pp. 142–153) is readily proved by the method employed by me (§ 12) for the case where $m = n$. The operation

$$\mathrm{col}_r + \omega\,\mathrm{col}_{m+r} + \omega^2\,\mathrm{col}_{2m+r} + \dots, \quad (r = 1, 2, \dots, m),$$

where ω is a prime nth root of 1, gives a determinant which is such that, by subtracting from each of its last $(n-1)m$ rows ω times another row, every element in the first m columns of these rows is made 0: consequently the minor occupying the first m rows and first m columns is a factor of the determinant.

Capetown, South Africa.
5th Nov. 1916.

AN ADDITIONAL NOTE ON CERTAIN EQUIMOMENTAL SYSTEMS OF PARTICLES.

By *Eric H. Neville.*

HAVING occasion a few days ago to repeat the canon that in the proof of a symmetrical result unsymmetrical work is seldom if ever the best, I was first shocked to remember that my treatment of a few questions on equimomental systems in a recent note* was unsymmetrical, and then glad to find that a symmetrical treatment of the same questions by the same method gave the familiar results even more directly.

A uniform triangle is equimomental with a set of equal particles at the mid-points of its sides; it follows that if the four faces of a tetrahedron are given uniform densities such that their four masses are equal, the whole surface of the tetrahedron is equimomental with a set of six equal particles at the mid-points of the edges. Hence a uniform tetrahedron with centroid G is equimomental with a distribution along the three lines joining the mid-points of opposite edges, the masses of the three lines being equal and the line-density in each being proportional to the square of distance from the centroid G, which is also the centroid of each line; the ratio of the square of the radius of gyration of such a line to the square of the half-length of the line is

$$\int_0^1 t^4 dt \Big/ \int_0^1 t^2 dt,$$

that is, $3/5$, and therefore if the mass of the line is m the line is equimomental with the system obtained by placing a particle of mass $3m/10$ at each end and the residue $2m/5$ at the centre. Hence a uniform tetrahedron of mass M is equimomental with a system formed of six particles of mass $M/10$ at the mid-points of the edges and a seventh of mass $2M/5$ at the centroid.

The system equimomental with a triangle, on which this proof depends, can be found by the same method. A uniform line of mass m is equimomental with the system composed of a particle of mass $m/6$ at each end and the residue $2m/3$ at the centre. Hence a *hollow* triangle ABC of mass m whose sides have such uniform line-densities that their masses are

* *Messenger of Mathematics*, vol. xlv. (1915), p. 94.

equal is equimomental with a system of three particles of mass $m/9$ at the vertices and three of mass $2m/9$ at the mid-points of the sides; this system is equimomental both with the set composed of particles of mass $2m/3$ at the mid-points of the sides with the negative residue $-m$ at the centroid G, and with the set composed of particles of mass $2m/3$ mid-way between the centroid and the vertices with the same residue at the centroid, and therefore is equimomental also with the system composed of six particles of mass $m/3$ at the mid-points of BC, CA, AB, GA, GB, GC, together with the residue $-m$ at G. It follows that a triangle ABC of uniform surface-density and mass M is equimomental with a distribution along the medians, together with a residue at the centroid; each median has total mass $2M/3$, distributed from the mid-point of the side to the point mid-way between G and the vertex with a line-density proportional to absolute distance from G; the ratio of the square of the radius of gyration to the square of the half-length of the occupied portion is

$$\int_0^1 t^3\,dt \Big/ \int_0^1 t\,dt,$$

that is, $1/2$, and therefore each line is equimomental with particles of one quarter of its total mass at the end-points of the occupied portion with the residue at G. Hence the triangle is equimomental with the system formed of particles of mass $M/6$ at the mid-points of BC, CA, AB, GA, GB, GC, for since the sum of the masses of these particles is M there is no residue; the set of particles of mass $M/6$ at the mid-points of GA, GB, GC is equimomental with the set of particles of mass $M/6$ at the mid-points of BC, CA, AB. Hence the triangle is equimomental both with the set of particles of mass $M/3$ at the mid-points of the sides and with the set of particles of mass $M/3$ mid-way between the centroid and the vertices; the latter of these sets may be replaced by a set of particles of mass $M/12$ at the vertices together with the residue $2M/3$ at the centroid.

In connection with the former note I have been reminded that the results follow almost without either geometry or analysis if integrals of inertia are treated as Dirichlet integrals and the necessary values are quoted; it is not suggested that a brief elementary method is therefore without interest.

ON CLASS RELATION FORMULÆ.

By *L. J. Mordell*, Birkbeck College, London.

§ 1. When Kronecker* in 1857 introduced his class relation formulæ, a new development, with far reaching consequences, was made in the Theory of Numbers. Concerning these, H. J. S. Smith† says, "They form a memorable accession to our knowledge of quadratic forms, and have opened an entirely new field of arithmetical enquiry. Their demonstration requires considerations of a very complicated kind; and as they are certainly among the most interesting, so also must they be reckoned among the most abstruse of arithmetical truths."

For the enumeration of some of these formulæ, taking a definite binary quadratic in the form $ax^2+2hxy+by^2$ with determinant $(-n)$, let $F(n)$ be the number of uneven classes, and $G(n)$ the whole number of classes of forms of determinant $-n$, the classes $(a, 0, a)$, $(2a, a, 2a)$ being counted as $\frac{1}{2}$ and $\frac{1}{3}$ respectively; to $F(0)$ we attribute the value 0, to $G(0)$ the value $-1/12$. If, however, the quadratic is taken in the form $ax^2+hxy+by^2$ with discriminant $(-n)$, let $H(n)$ be the number of classes of forms of discriminant $-n$; the classes $(a, 0, a)$, (a, a, a) being counted as $\frac{1}{2}$ and $\frac{1}{3}$ respectively, to $H(0)$ we attribute the value $-1/12$. Then

$$F(n)+2F(n-1^2)+2F(n-2^2)+\ldots=\Delta(n)-\Gamma(n)\ldots(1),$$

$$F(n)-2F(n-1^2)+2F(n-2^2)-\ldots=\Gamma'(n)\ldots\ldots\ldots\ldots(2),$$

the left-hand sides being continued so long as $n-k^2$, $k=0$, ±1, ±2, ... is not negative.‡ Also $\Gamma(n)$ represents the sum of those divisors of n which do not surpass $\sqrt{n}$, and which are of the same parity as their conjugate divisors; $\Gamma'(n)$ represents the sum of the same divisors, each divisor being taken positively or negatively according as the sum of itself and its conjugate divisor $\equiv 2$ or 0, mod. 4. If n is a perfect square we are to replace $\sqrt{n}$ by $\frac{1}{2}\sqrt{n}$ in the sums $\Gamma(n)$

* *Monatsberichte* (1857) or *Crelle*, vol. lxvii. (1860).

† *Collected Works*, vol. i., page 321.

‡ This convention is usual in class relation formulæ, and is used throughout this paper.

and $\Gamma'(n)$. Finally $\Delta(n)$ represents the sum of those divisors of n whose conjugates are uneven.

Kronecker proved his formulæ originally by means of the modules which admit of complex multiplication, and a similar method was adopted by Joubert, Hermite, and Smith, who proved some of Kronecker's formulæ thus in his *Report on the Theory of Numbers*. The method can be seen in its simplest form in the proof of the equation*

$$H(4n) + 2H(4n-1^2) + 2H(4n-2^2) + \ldots = \Phi(n) + \Psi(n),$$

where $\Phi(n)$ is the sum of the divisors of n, and $\Psi(n)$ represents the excess of the sum of those divisors of n which are $> \sqrt{n}$ over the sum of those which are $< \sqrt{n}$.

But Kronecker had also noticed that they could be proved in a comparatively elementary way by the multiplication of different expansions of theta functions, and published a note to that effect (after some of his results had been proved in this way by Hermite†). Thus‡ by expanding $H^2H_1\cos x/\Theta^2$, where Θ, H, &c., are the Jacobian theta-functions, as the product of

$$\frac{H}{\Theta}\cos x \text{ and } \frac{HH_1}{\Theta}, \text{ or of } \frac{H^2}{\Theta^2} \text{ and } H_1\cos x,$$

$$\text{or of } \frac{HH_1}{\Theta^2} \text{ and } H\cos x,$$

he found that

$$\frac{\theta_1\eta_1^2}{8\pi q^{\frac{1}{4}}}\int_0^{\pi}\frac{H^2H_1}{\Theta^2}\cos x\,dx = \sum_1^{\infty} F(n)\,q^n$$

where $\theta_1 = \Theta_1(0)$, $\eta_1 = H_1(0)$, and so proved equations (1, 2). But when a particular class-relation formula is known it is no easy matter to see *a priori* what is the function to be expanded.

A large number of expansions of functions such as HH_1/Θ, etc., are given by Hermite (*loc. cit.*), and his papers are the origin of the modern theory of doubly periodic functions of the second and third kinds. Kronecker's results also proved to be a powerful stimulus to the study of the Theory of Functions, Groups, Modular Functions, Automorphic Functions, and of Bilinear Forms. This is an extremely good

* Weber, *Lehrbuch der Algebra*, vol. iii., p. 423; Klein-Fricke, *Modul-functionen*, vol. 2, p. 174.

† *Collected Works*, vol. ii., p. 109; or *Liouville*, 2e ser. t. 7 (1862), p. 25; *Collected Works*, vol. ii., p. 240, or *Liouville*, 2e, ser. t. 9 (1864), p. 145.

‡ *Monatsberichte* (May, 1862), pp. 307, 308.

illustration of the remarks* due to Gauss. 'Gauss,' we are told by his biographer, 'held mathematics to be the Queen of the Sciences, and Arithmetic to be the Queen of Mathematics.' 'She sometimes condescends'—so spoke the great Astronomer and Physicist—'to render services to Astronomy and the other natural sciences, but under all circumstances the first place is her due.' In a more serious mood he wrote: 'The higher arithmetic presents us with an inexhaustible storehouse of interesting truths—of truths, too, which are not isolated, but stand in the closest relation to one another, and between which, with each successive advance of the science, we continually discover new and sometimes wholly unexpected points of contact.'

Kronecker's results were considerably extended in fairly recent times by Gierster† and Hurwitz‡ by means of considerations depending upon the modular correspondences, and one may look upon the treatise by Klein and Fricke on the Modular Functions as leading up to the general results due to Hurwitz, and giving an insight into their proper meaning. He found amongst other results that

$$\sum_{k \equiv \alpha \bmod. p} H(4n - k^2) = A \quad (k^2 \leq 4n),$$

the summation extending to positive, negative, and zero integer values of $k \equiv \alpha \bmod. p$ { α is given, p is an odd prime}, and A is expressed in terms of X_β {the sum of those divisors of n which are $< n$ and $\equiv \pm \beta \bmod. p$} and of $\psi(n)$, $\chi(n)$, etc., the coefficients of $r^n p$ in the expansions of $j_1(\omega)$, $j_2(\omega)$, etc., the integrals of the first kind for the fundamental polygon defined by the linear fractional group of order p.

But other results have been given, as for example the one by Liouville§

$$F(4m - 1^2) - 3F(4m - 3^2) + 5F(4m - 5^2) - \ldots = \Sigma(a^2 - 4b^2),$$

where m is an odd number, and the right-hand side summation refers to all the solutions of $m = a^2 + 4b^2$, in which a is positive; and if no solutions exist, the right-hand sum is replaced by zero. This, it would seem, is extremely difficult to deduce from the modular correspondence theory, though it can be

* H. J. S. Smith, *Proceedings of the London Mathematical Society*, vol. viii., or *Collected Works*, vol. ii., p. 169.

† *Math. Annalen*, vol. xvii., 1880; vol. xxi., 1883.

‡ *Math. Annalen*, vol. xxv., 1885; Leipz, *Berichten*, May, 1885; or Klein-Fricke, *Modul-functionen*, vol. ii., p. 637.

§ Liouville, vol. xvi., 2nd series (1869).

proved quite simply by the multiplication of various series. Stieltjes.* however, claims to have proved this and similar ones arithmetically, but his demonstration was never published.

Coming back to the results given by Hurwitz, it seemed to me that more light might be thrown upon the subject by considering the generating functions of such series as $\sum_{n=1}^{\infty} X_\beta q^n$, and I was finally led to the following extremely simple and direct method for dealing with such questions.†

It dependsupon defining an integral, transcendental function of x by means of the equations

$$f(x+\omega)+f(x)=\theta_{00}(x),$$
$$f(x+1)=f(x).$$

An expansion is then found for $f(x)/\theta_{11}(x)$ by contour integration, and it is shewn that

$$f'(0)/\theta_{00}=-4\iota\pi\sum_{M=1}^{\infty}(-1)^M F(M)q^M,$$

where $$f'(0)=\left[\frac{df(x)}{dx}\right]_{x=0}.$$

But the function $f(x)$ lends itself very readily to the finding of various expansions, for example

$$\frac{f(x)}{\theta_{00}(x)}\theta_{gh}(x+\xi),\ \frac{f(x)}{\theta_{00}(x)}\theta_{00}(2x,\ 2\omega),$$
$$\frac{f(x)}{\theta_{00}(x)}\theta_{01}(2x,\ 2\omega),\ \frac{f(x)}{\theta_{00}(x)}\theta_{00}(3x,\ 3\omega).$$

Differentiating with respect to x, and then putting $x=0$, we find various class relation formulæ. The formulæ for

$$\frac{f(x)}{\theta_{00}(x)}\theta_{00}(mx,\ m\omega)$$

are more complicated.

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* *Comptes Rendus*, 1883, or *Collected Works*, vol. i.

† Since writing this paper I have become acquainted with a paper by G. Humbert, "Nombres de classes des formes quadratiques," in Liouville, 1907. He has obtained a large number of interesting results by the application of Hermite's classical method, and in particular my result (f), which seems to be due to Petr, *Acad. des Sciences de Bohême*, 1900–01. Both of these authors have also given class relation formulæ involving the representations of numbers by indefinite quadratic forms. Humbert has given a formula for

$$F(m)-2F(m-2.1^2)+2F(m-2.2^2)\ \ldots\ ,$$

which, while different from my result, has a great resemblance to it.

If $F(x)$ be defined similarly by means of

$$F(x) - F(x+\omega) = \theta_{10}(x),$$

$$F(x+1) = -F(x),$$

it is found that

$$F(0)/\theta_{10} = 2\iota\pi \sum_{M=0}^{\infty} [4F(M) - 3G(M)]q^M,$$

and again $F(x)$ is very useful for finding expansions.

§ 2. Consider the integral transcendental function of x defined by the series

$$f(x) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi ix}/(1+q^{2n}),$$

where $q = e^{\iota\pi\omega}$ and $I(\omega) > 0$, so that as usual $|q| < 1$. Some of its obvious properties are

$$f(x) + f(x+\omega) = \theta_{00}(x) \ldots\ldots\ldots\ldots\ldots (a),$$

$$f(x+1) = f(x),$$

$$f(-x) = f(x+\omega) \ldots\ldots\ldots\ldots (b),$$

$$2f(0) = \theta_{00} \ldots\ldots\ldots\ldots\ldots\ldots (c).$$

The value of $f[\frac{1}{2}(1+\omega)]$ is rather useful, and we find it as follows. Consider the integral

$$\int \frac{\sec \pi (t-x)}{\theta_{11}(t)}\, dt$$

taken around the contour formed by the parallelogram $ABCD$ whose angular points are $\frac{1}{2} + (n+\frac{1}{2})\omega$, $-\frac{1}{2} + (n+\frac{1}{2})\omega$, $-\frac{1}{2} - (n+\frac{1}{2})\omega$, $\frac{1}{2} - (n+\frac{1}{2})\omega$, n a positive integer. Since the integrand admits unity as a period, the integrals along DA and BC cancel each other, and when $n \to \infty$, remembering that

$$\theta_{11}(t + n\omega) = (-1)^n q^{-n^2} e^{-2n\pi\iota t}\theta_{11}(t),$$

it is easily seen that the integrals along AB and CD each tend to zero.* But assuming that x is not of the form $a + b\omega$ $\{a,\ b$ integers$\}$, and that its real part lies between

* This is more fully gone into in a similar case in § 6.

0 and -1, the only singularities of the integrand within the contour are simple poles at $t = x + \frac{1}{2}$, with a residue $-1/\pi\theta_{10}(x)$, and at $t = 0, \pm\omega, \pm 2\omega, \ldots, \pm n\omega$. But the residue at $t = r\omega$ is $\sec\pi(r\omega - x)(-1)^r q^{r^2}/\theta_{11}'$. Hence

$$\theta_{11}'/\pi\theta_{10}(x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \sec\pi(n\omega - x),$$

an expansion really valid over the whole of the x plane.

Putting $x = 0$, and noticing that $\theta_{11}' = \pi\theta_{00}\theta_{01}\theta_{10}$, we have

$$\theta_{00}\theta_{01} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \sec(n\pi\omega),$$

or

$$\theta_{00}\theta_{01} = 2\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n}/(1+q^{2n}),$$

$$= 2f\left[\tfrac{1}{2}(1+\omega)\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (d).$$

From equation (a), since $\theta_{00}(x + n\omega) = q^{-n^2}e^{-2n\pi ix}\theta_{00}(x)$, we find, when n is a positive integer,

$$f(x) + (-1)^{n+1} f(x + n\omega)$$

$$= \theta_{00}(x)\left[1 - q^{-1}e^{-2\pi ix} + q^{-4}e^{-4\pi ix} \ldots \pm q^{-(n-1)^2}e^{-2\pi i(n-1)x}\right] \ldots (e).$$

Putting $x = 0$, then since $2f(0) = \theta_{00}$, we find

$$2(-1)^{n+1} f(n\omega) = \theta_{00}\left[1 - 2q^{-1} + 2q^{-4} \ldots \pm 2q^{-(n-1)^2}\right]$$

$$= \theta_{00} R_n, \text{ say} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (f).$$

And, since $f(-x) = f(x + \omega)$,

$$2(-1)^n f(-n\omega) = \theta_{00} R_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots (g).$$

Evidently for all integer values of n

$$f\left(\frac{1+\omega}{2} + n\omega\right) = (-1)^n f\left(\frac{1+\omega}{2}\right) = (-1)^n \theta_{00}\theta_{01}/2,$$

since

$$\theta_{00}\left(\frac{1+\omega}{2}\right) = 0.$$

Having established these preliminary considerations, consider the integral

$$\int \frac{f(t)\operatorname{cosec}\pi(t-x)}{\theta_{11}(t)}\, dt$$

taken around the parallelogram $ABCD$. As before, since the integrand admits of the period unity, the integrals along DA and BC cancel each other, and noting equation (e) we can show that when $n \to \infty$ the integrals along AB and CD each tend to zero.* The residue of the integrand at $t = x$ is $f(x)/\pi\theta_{11}(x)$; the residue at $t = r\omega$ is

$$f(r\omega)\operatorname{cosec}\pi(r\omega - x)(-1)^r q^{r^2}/\theta'_{11},$$

hence

$$f(x)/\pi\theta_{11}(x) + \sum_{n=-\infty}^{\infty} f(n\omega)(-1)^n q^{n^2}\operatorname{cosec}\pi(n\omega - x)/\theta'_{11} = 0.$$

When n is negative in this summation, writing $-n$ for n and noting equations f, g, we find that

$$\theta'_{11} f(x)/\pi\theta_{00}\theta_{11}(x)$$

$$= \sum_{n=0}^{\infty} q^{n^2} R_n \operatorname{cosec}\pi(n\omega - x) + \sum_{n=1}^{\infty} q^{n^2} R_{n+1} \operatorname{cosec}\pi(n\omega + x),$$

an expansion really valid over all the x plane.

Multiplying both sides by $\theta_{11}(x)$, differentiating and putting $x = 0$, we find

$$2f'(0)/\pi\theta_{00} = \sum_{n=1}^{\infty} q^{n^2} R_n \operatorname{cosec} n\pi\omega + \sum_{n=1}^{\infty} q^{n^2} R_{n+1} \operatorname{cosec} n\pi\omega$$

$$= 2\sum_{n=1}^{\infty} q^{n^2} \operatorname{cosec} n\pi\omega\,(1 - 2q^{-1} + 2q^{-4} + \ldots \pm 2q^{-(n-1)^2} \mp q^{-n^2}),$$

so that

$$-f'(0)/2\iota\pi\theta_{00} = \sum_{n=1}^{\infty} q^{n^2+n}(1 - 2q + \ldots \pm 2q^{-(n-1)^2} \mp q^{-n^2})/(1 - q^{2n})$$

$$= \sum_{n=1}^{\infty} (-1)^r q^{n^2 - r^2 + n(2t+1)}$$

$$r = 0, \pm 1, \ldots \pm (n-1), n$$

$$t = 0, 1, 2, \ldots,$$

$$= 2\sum_{M=1}^{\infty} (-1)^M F(M) q^M \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (g'),$$

as will be shewn in a moment. Hence, substituting for $f'(0)$, we have

$$-\sum_{n=-\infty}^{\infty} n q^{n^2}/\theta_{00}(1 + q^{2n}) = 2\sum_{M=1}^{\infty} (-1)^M F(M) q^M,$$

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* Cf. § 6.

which can be written as

$$\sum_{n=1}^{\infty} nq^{n^2}(1-q^{2n})/(1+q^{2n}) = 2\theta_{00}\sum_{M=1}^{\infty}(-1)^{M+1}F(M)\,q^M.$$

Changing q into $-q$, and equating coefficients of q^m, since $\theta_{01}=1-2q+2q^4-\ldots$, we find*

$$F(m)-2F(m-1^2)+2F(m-2^2)-\ldots=\Sigma a\,(-1)^{(a+d)/2+1},$$

where a refers to all divisors of m which are $\leq\sqrt{(m)}$, and of the same parity as their conjugate divisors d. But if $a=\sqrt{(m)}$, we replace the coefficient a by $\frac{1}{2}a$. This is Kronecker's result (2).

§ 3. We shall now show that the number of solutions† of

$$M=n^2-r^2+n\,(2t+1),$$

$$n=1,\ 2,\ \ldots,$$

$$r=0,\ \pm1,\ \pm2,\ \ldots,\ \pm(n-1),\ n$$

$$t=0,\ 1,\ 2,\ \ldots,$$

is $2F(M)$ or twice the number of classes of uneven forms of determinant $-M$. We notice that corresponding to each set of values for n, r, t we have the quadratic form

$$nx^2+2rxy+(n+2t+1)\,y^2$$

of determinant $-M$. First let n be odd. Then when r has the values $0, \pm1, \pm2, \ldots, \pm\frac{1}{2}(n-1)$ the number of solutions is equal to the number of the reduced non-ambiguous forms whose first coefficient is odd and whose third coefficient is even, the number of which may be represented by $(0, e)$, together with the number of ambiguous forms of the type $(n, 0, n+2t+1)$. When r has the values $\frac{1}{2}(n+1), \ldots, n$, we write $x-y$ for x and consider the form

$$\{n, -(n-r),\ 2n-2r+2t+1\};$$

and when r has the values $-\frac{1}{2}(n+1), \ldots, -(n-1)$, we write

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* The expansion of $f'(x)/\theta_{00}(x)$ suggested the simple arithmetical proof of this formula that I gave in the *Messenger of Mathematics*, vol. xlv. (1916), pp. 177—180.

† G. Humbert, in *Liouville* (1907), has found the number of solutions of some equations of this kind very simply by means of the modular division of the plane.

$x+y$ for x and consider the form $(n,\ n+r,\ 2n+2r+2t+1)$. Putting $-k=n-r$ in the first form and $k=n+r$ in the second, we have the forms

$$(n,\ k,\ 2k+2t+1),\ k=0,\ \pm1,\ \pm2,\ \ldots,\ \pm\tfrac{1}{2}(n-1).$$

Hence each reduced non-ambiguous form $(a,\ b,\ c)$, whose first and third coefficients are odd, supplies us with two sets of solutions for $n,\ k,\ t$, according as we take $a,\ b,\ c$ equal to $n,\ k,\ 2k+2t+1$ or $2k+2t+1,\ k,\ n$. The number of these solutions may be denoted by $2\,(0,\ 0)$. To this must be added the solutions when $k=0$, *i.e.* the number of ambiguous forms of the type $(n,\ 0,\ 2t+1)$, and also those for which

$$n=2k+2t+1,\ k\neq0,$$

i.e. twice the number of ambiguous forms of the type $(n,\ r,\ n)$, $r=1,\ 2,\ \ldots,\ \frac{1}{2}(n-1)$, since $n=2k+2t+1$, with

$$k=\pm1,\ \pm2,\ \ldots.\ \pm\tfrac{1}{2}(n-1),$$

does not give a negative value for t.

When n is even we carry out a similar process, and we arrange the solutions in the form

$$n,\ r,\ n\ +2t+1,\ r=0,\ \pm1,\ \ldots,\ \pm\tfrac{1}{2}(n-2),$$

$$n,\ k,\ 2k+2t+1,\ k=0,\ \pm1,\ \pm2,\ \ldots,\ \pm\tfrac{1}{2}n,$$

and we find the number of the first to be $(e,\ 0)$ plus the number of ambiguous forms of the type $(n,\ 0,\ n+2t+1)$. The number of the second is $(e,\ 0)+(0,\ e)$ plus the number of ambiguous forms of the type $(n,\ 0,\ 2t+1)$ plus twice the number of the type $(n,\ \frac{1}{2}n,\ n+2t+1)$. The total number of solutions is evidently $2\,\{(0,\ 0)+(0,\ e)+(e,\ 0)\}$ plus twice the number of forms of the type

$$(2n+1,\ r,\ 2n+1),\ r=\pm1,\ \pm2,\ \ldots,\ \pm n,$$

or say

$$2\,(2n+1,\ r,\ 2n+1),$$

$$\text{plus}\quad(2n+1,\ 0,\ 2n+2t+2),$$

$$\text{plus}\quad(2n+1,\ 0,\ 2t+1),$$

$$\text{plus}\quad(2n,\ 0,\ 2n+2t+1),$$

$$\text{plus}\quad(2n,\ 0,\ 2t+1).$$

$$\text{plus}\ 2\,(2n,\ n,\ 2n+2t+1),$$

which evidently gives $2F(M)$ if the form $(a,\ 0,\ a)$ is reckoned as $\frac{1}{2}$.

§ 4. Consider now the function defined by

$$\phi(x) = \frac{f(x)}{\theta_{00}(x)}\,\theta_{g,h}(x+\xi),$$

where $\theta_{g,h}$ is a theta-function of the first order, with characteristic g, h, whose expansion may be taken as $\sum_{n=\infty}^{-\infty} A_n e^{n\pi i x}$; and ξ is an arbitrary constant.

Now

$$\phi(x+\omega) = (-1)^h \frac{f(x+\omega)}{\theta_{00}(x)}\,\theta_{g,h}(x+\xi)\,e^{-2\pi i\xi},$$

and since $$f(x)+f(x+\omega) = \theta_{00}(x),$$

$$\left.\begin{aligned} &\phi(x)+(-1)^h e^{2\pi i\xi}\phi(x+\omega) = \theta_{g,h}(x+\xi) = \sum_{n=\infty}^{-\infty} A_n e^{n\pi i(x+\xi)} \\ &\text{also} \qquad \phi(x+1) = (-1)^g \phi(x) \end{aligned}\right\} \ldots(h).$$

The only singularities of $\phi(x)$ are simple poles at the zeros of $\theta_{00}(x)$, *i.e.* $x = \frac{1}{2}(1+\omega)+a+b\omega$, a and b integers, and for these values of x we have, from equation (a),

$$f(x) = (-1)^b f\{\tfrac{1}{2}(1+\omega)\},$$

so that the residues of $\phi(x)$ and of

$$(-1)^b f\{\tfrac{1}{2}(1+\omega)\}\,\theta_{g,h}(x+\xi)/\theta_{00}(x)$$

at the points $x = \frac{1}{2}(1+\omega)+a+b\omega$ are equal.

These conditions, however, suffice (in general) to define completely the uniform, transcendental function $\phi(x)$; for if there were two such functions, their difference, by equations (h), would be a doubly periodic function of the second kind, with no singularities in a fundamental parallelogram.

Hence we can write

$$\phi(x) = \sum_{n=\infty}^{-\infty} \frac{A_n e^{n\pi i(x+\xi)}}{1+(-1)^h q^n e^{2\pi i\xi}} + A(x),$$

so that equations (h) will be satisfied if

$$\left.\begin{aligned} &A(x+1) = (-1)^g A(x) \\ &A(x+\omega) = (-1)^{h+1} e^{-2\pi i\xi} A(x) \end{aligned}\right\} \ldots\ldots\ldots(i).$$

Moreover, $A(x)$ must have simple poles at the zeros of $\theta_{00}(x)$, with the same residues there that $\phi(x)$ has. Hence $A(x)$ is a doubly periodic function of the second kind, and we have

$$A(x) = f\left[\tfrac{1}{2}(1+\omega)\right] \frac{\theta_{g,h}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}}{\theta_{g,h+1}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}} \cdot \frac{\theta_{g,h+1}(x+\xi)}{\theta_{00}(x)};$$

for this evidently satisfies equations (*i*), and the residue at the simple pole $x = \frac{1}{2}(1+\omega) + a + b\omega$ is equal to the residue of $\phi(x)$ at the same place.

Hence, since

$$2f\left\{\tfrac{1}{2}(1+\omega)\right\} = \theta_{00}\theta_{01},$$

$$\frac{f(x)}{\theta_{00}(x)}\theta_{g,h}(x+\xi) = \sum_{n=\infty}^{-\infty} \frac{A_n e^{n\pi\iota(x+\xi)}}{1+(-1)^h q^n e^{2\pi\iota\xi}} + \frac{\theta_{00}\theta_{01}}{2} \frac{\theta_{g,h}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}}{\theta_{g,h+1}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}} \cdot \frac{\theta_{g,h+1}(x+\xi)}{\theta_{00}(x)},$$

which holds for all values of x and ξ.

Differentiating this expansion with respect to x, and then putting $x=0$, we find*

$$\frac{f(0)}{\theta_{00}}\theta'_{g,h}(\xi) + \frac{f'(0)}{\theta_{00}}\theta_{g,h}(\xi) = \sum_{n=\infty}^{-\infty}{}' \frac{n\pi\iota A_n e^{n\pi\iota\xi}}{1+(-1)^h q^n e^{2\pi\iota\xi}} + \frac{\theta_{01}}{2} \frac{\theta_{g,h}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}}{\theta_{g,h+1}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}} \theta'_{g,h+1}(\xi).$$

In particular, taking $g, h = 0, 0$, we have

$$\frac{f(0)}{\theta_{00}}\theta'_{00}(\xi) + \frac{f'(0)}{\theta_{00}}\theta_{00}(\xi) = \sum_{n=\infty}^{-\infty} \frac{2n\pi\iota\, q^{n^2} e^{2n\pi\iota\xi}}{1+q^{2n}e^{2\pi\iota\xi}} + \iota\frac{\theta_{01}}{2}\frac{\theta_{11}(\xi)}{\theta_{10}(\xi)}\theta_{01}(\xi),$$

noting the values of $\theta_{00}\left\{\tfrac{1}{2}(1+\omega)+\xi\right\}$, &c. Writing now $-\xi$ for ξ, and adding the expansion so found to the one above, we have

$$\frac{f'(0)}{\theta_{00}}\theta_{00}(\xi) = \sum_{n=\infty}^{-\infty}{}' \frac{n\pi\iota q^{n^2} e^{2n\pi\iota\xi}}{1+q^{2n}e^{2\pi\iota\xi}} + \sum_{n=\infty}^{-\infty}{}' \frac{n\pi\iota q^{n^2} e^{-2n\pi\iota\xi}}{1+q^{2n}e^{-2\pi\iota\xi}} + \iota\frac{\theta_{01}}{2}\frac{\theta_{11}(\xi)}{\theta_{10}(\xi)}\theta'_{01}(\xi).$$

* Σ' denotes that $n=0$ is to be omitted from the summation.

But $$f'(0)/\theta_{00} = 4\pi\iota \sum_{M=1}^{\infty} (-1)^{M+1} F(M)\, q^M,$$

and, writing $-n$ for n in the second summation, we have

$$8\theta_{00}(\xi) \sum_{M=1}^{\infty} (-1)^{M+1} F(M)\, q^M$$

$$= \sum_{n=\infty}^{-\infty}{}' 2n\, q^{n^2} e^{2n\pi\iota\xi} \left(\frac{1 - q^{2n} e^{2\pi\iota\xi}}{1 + q^{2n} e^{2\pi\iota\xi}}\right) + \frac{\theta_{01}}{\pi} \frac{\theta_{11}(\xi)}{\theta_{10}(\xi)} \theta_{01}'(\xi),$$

which, by changing q into $-q$ and ξ into $\xi + \frac{1}{2}$, can be written as

$$8\theta_{00}(\xi) \sum_{M=1}^{\infty} F(M) q^M = -\sum_{n=\infty}^{-\infty}{}' 2nq^{n^2} e^{2n\pi\iota\xi} \left(\frac{1 + q^{2n} e^{2\pi\iota\xi}}{1 - q^{2n} e^{2\pi\iota\xi}}\right)$$

$$+ \frac{\theta_{00}}{\pi} \frac{\theta_{10}(\xi)}{\theta_{11}(\xi)} \theta_{01}'(\xi) \ldots\ldots (j).$$

Putting $\xi = 0$, we have

$$8\theta_{00} \sum_{M=1}^{\infty} F(M) q^M = -4 \sum_{n=1}^{\infty} nq^{n^2} \left(\frac{1 + q^{2n}}{1 - q^{2n}}\right) + \frac{1}{\pi} \frac{\theta_{10}\theta_{00}\theta_{01}''}{\theta_{11}'}.$$

The second term on the right-hand side can be written as

$$\frac{1}{\pi^2} \frac{\theta_{01}''}{\theta_{01}} = \frac{1}{\pi^2} \frac{d}{dx} \left(\frac{\theta_{01}'(x)}{\theta_{01}(x)}\right) \text{ when } x = 0,$$

and this is found to be

$$8 \sum_{n=1}^{\infty} q^{2n-1}/(1 - q^{2n-1})^2,$$

so that, equating coefficients of q^m, we find

$$F(m) + 2F(m - 1^2) + 2F(m - 2^2) + \ldots = -\Sigma a + \Sigma b,$$

where b refers to all the divisors of m whose conjugate divisors are odd. This is Kronecker's result (1).

§ 5. We can also find other expansions involving $f(x)$. For putting

$$\chi(x) = \frac{f(x)}{\theta_{00}(x)} \theta_{01}(2x,\ 2\omega),$$

we have

$$\chi(x+\omega) = -e^{-\pi i(2x+\omega)} \frac{f(x+\omega)}{\theta_{00}(x)} \theta_{01}(2x, 2\omega).$$

Hence

$$\chi(x) - e^{\pi i(2x+\omega)} \chi(x+\omega) = \theta_{01}(2x, 2\omega) = \sum_{n=\pm\infty} A_n e^{2n\pi ix}, \text{ say...}(k),$$

and

$$\chi(x+1) = \chi(x).$$

Moreover, since the zeros of $\theta_{00}(x)$ are also zeros of $\theta_{01}(2x, 2\omega)$, $\chi(x)$ is a transcendental, integral function of x, and we can write

$$\chi(x) = A\theta_{00}(x) + \sum_{n=\pm\infty} B_n e^{2n\pi ix},$$

where A is independent of x, and the B's are constants which, from equation (k), satisfy the difference equation

$$B_n - q^{2n-1} B_{n-1} = A_n.$$

Because of the constant A we need only take a particular solution of this equation, and we write, if n is positive,

$$B_n = q^{n^2}[q^{-n^2} A_n + q^{-(n-1)^2} A_{n-1} + \ldots + A_0],$$

$$B_{-n} = -q^{n^2}[q^{-(n-1)^2} A_{-(n-1)} + q^{-(n-2)^2} A_{-(n-2)} + \ldots + q^{-1} A_{-1}].$$

But since, $A_{2r} = (-1)^r q^{2r^2}$ and $A_{2r+1} = 0$,

$$B_n = \sum_{r=0}^{n} q^{n^2-r^2} A_r = \sum_{r=0}^{r\leq n|2} (-1)^r q^{n^2-2r^2},$$

and

$$B_{-n} = -\sum_{r=1}^{r\leq (n-1)|2} (-1)^r q^{n^2-2r^2}.$$

Hence

$$\frac{f(x)}{\theta_{00}(x)} \theta_{01}(2x, 2\omega) = A\theta_{00}(x) + \sum_{n=\pm\infty} B_n e^{2n\pi ix} \ldots\ldots(l).$$

Writing now $-x$ for x we have

$$\frac{f(-x)}{\theta_{00}(x)} \theta_{01}(2x, 2\omega) = A\theta_{00}(x) + \sum_{n=\pm\infty} B_{-n} e^{2n\pi ix}.$$

Adding these equations and remembering that

$$f(x)+f(-x)=\theta_{00}(x),$$

we have

$$\theta_{01}(2x,\ 2\omega)=2A\theta_{00}(x)+\sum_{n=\pm\infty}(B_n+B_{-n})e^{2n\pi\iota x},$$

giving us at once $2A+1=0$ {noting the value of B_n}.*

Differentiating equation (l), and putting $x=0$, we have

$$\frac{f'(0)}{\theta_{00}}\theta_{01}(0,\ 2\omega)=2\pi\iota\sum_{n=\pm\infty}nB_n$$

$$=2\pi\iota\sum_{n=0}^{\infty}n(B_n-B_{-n}).$$

Hence from equation (g')

$$4\iota\pi\theta_{01}(0,\ 2\omega)\left[F(1)q-F(2)q^2+F(3)q^3-\ldots\right]$$

$$=2\pi\iota\sum_{n=1}^{\infty}\sum_{r\le n/2}^{r\ge-(n-1)/2}n(-1)^r q^{n^2-2r^2}.$$

Equating coefficients of q^m, we find

$$F(m)-2F(m-2.1^2)+2F(m-2.2^2)-2F(m-2.3^2)+\ldots$$

$$=\tfrac{1}{2}\Sigma x(-1)^{x+y+1},$$

where the summation extends to all values of x and y satisfying

$$x^2-2y^2=m$$

with $x>0$ and $-\frac{1}{2}(x-1)\le y\le\frac{1}{2}x$. Evidently the right-hand summation is replaced by zero if no such solutions exist.

We may notice that, in this example, we really do not require the value of A. If, however, $\theta_{01}(2x,\ 2\omega)$ had been replaced by $\theta_{01}(2nx,\ 2n\omega)$, n an integer, instead of one arbitrary constant A, we should have introduced $2n-1$ arbitrary constants. While there is no difficulty in finding them, I have not been able to reduce them to a sufficiently simple

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* If n is odd, $B_n+B_{-n}=q^{n^2}$, but if n is even, $B_n+B_{-n}=q^{n^2}+(-1)^{\frac{1}{2}n}q^{n^2/2}$.

form to be of use for deriving class relation formulæ. I may notice, however, a few other simple expansions.

$$\frac{f(x)}{\theta_{00}(x)}\theta_{00}(2x, 2\omega) = A'\theta_{01}(x) + \sum_{n=\pm\infty} B_n' e^{2n\pi\iota x}$$

$$+ f\left(\frac{1+\omega}{2}\right)\frac{\theta_{11}(2x, 2\omega)}{\theta_{00}(x)}\frac{\theta_{00}(1+\omega, 2\omega)}{\theta_{11}(1+\omega, 2\omega)}$$

$$= A'\theta_{01}(x) + \sum_{n=\pm\infty} B_n' e^{2n\pi\iota x}$$

$$+ \iota\frac{\theta_{00}\theta_{01}}{2}\frac{\theta_{11}(2x, 2\omega)}{\theta_{00}(x)}\frac{\theta_{10}(0, 2\omega)}{\theta_{01}(0, 2\omega)},$$

where
$$\left.\begin{aligned} B_n' &= (-1)^n \sum_{r=0}^{r<n^2} q^{n^2-2r^2} \\ B_{-n}' &= (-1)^{n+1} \sum_{r=1}^{r\leq(n-1)^2} q^{n^2-2r^2} \end{aligned}\right\} n \text{ positive.}$$

Differentiating, and putting $x = 0$, we find

$$4\iota\pi\,\theta_{00}(0, 2\omega)\,[F(1)q - F(2)q^2 + F(3)q^3 - \ldots]$$

$$= \sum_{n=1}^{\infty} 2n\pi\iota(B_n' - B_{-n}') + \iota\pi\theta_{01}\theta_{10}^2(0, 2\omega)\,\theta_{00}(0, 2\omega),$$

which, by changing ω into $\omega + 1$, becomes

$$2\theta_{00}(0, 2\omega)\,[F(1)q + F(2)q^2 + F(3)q^3 + \ldots]$$

$$= -\Sigma x q^m + \tfrac{1}{2}\theta_{00}\theta_{00}(0, 2\omega)\,\theta_{10}^2(0, 2\omega).*$$

Another expansion is given by

$$\frac{f(x)}{\theta_{00}(x)}\theta_{00}(3x, 3\omega) = A_1\theta_{01}(2x, 2\omega)$$

$$+ A_2\theta_{11}(2x, 2\omega) + \sum_{n=\pm\infty} B_n e^{2n\pi\iota x},$$

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* This can be expressed as a class relation formula by using Kronecker's result

$$4\sum_{0}^{\infty} F(4n+2)q^{\frac{1}{4}(4n+2)} = \theta_{10}^2\theta_{00},$$

a proof of which is given in my "Note on class relation formulæ," *Messenger of Mathematics*, vol. xlv. (1915), pp. 76–80.

where now, n being positive,

$$B_n = \sum_{r=0}^{r\leq n/3} (-1)^{(n-3r)/2} q^{\frac{1}{2}(n^2-3r^2)},$$

$$B_{-n} = -\sum_{r=1}^{r\leq (n-2)/3} (-1)^{(n-3r)/2} q^{\frac{1}{2}(n^2-3r^2)},$$

only those values of r within the assigned limits being taken, for which $r \equiv n \bmod. 2$.

Changing x into $x+\frac{1}{2}$, we have

$$\frac{f(x+\frac{1}{2})}{\theta_{01}(x)}\theta_{01}(3x, 3\omega) = A_1\theta_{01}(2x, 2\omega) - A_2\theta_{11}(2x, 2\omega) + \sum_{n=\pm\infty} (-1)^n B_n e^{2n\pi\iota x}.$$

Adding these formulæ, differentiating and putting $x=0$,

$$\frac{f'(0)}{\theta_{00}}\theta_{00}(0, 3\omega) + \frac{f'(\frac{1}{2})}{\theta_{01}}\theta_{01}(0, 3\omega) = \sum_{n=\pm\infty} 8n\pi\iota B_{2n}.$$

Since $\frac{f'(\frac{1}{2})}{\theta_{01}}$ is equal to what $\frac{f'(0)}{\theta_{00}}$ becomes when ω is replaced by $\omega+1$, this equality becomes

$$4\iota\pi\theta_{00}(0, 3\omega)[F(1)q - F(2)q^2 + F(3)q^3 - \ldots] + 4\iota\pi\theta_{01}(0, 3\omega)[-F(1)q - F(2)q^2 - F(3)q^3 - \ldots] = \sum_{n=0}^{\infty} 8n\pi\iota[B_{2n} - B_{-2n}],$$

and we can write

$$B_{2n} = \sum_{r=0}^{r\leq n/3} (-1)^{n+r} q^{2(n^2-3r^2)},$$

$$B_{-2n} = -\sum_{r=1}^{r\leq (n-1)/3} (-1)^{n+r} q^{2(n^2-3r^2)}.$$

Hence we have*

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* We can show very easily that $2A_1 + 1 = 0$. A_2 can be found under different forms by giving special values to x; or by multiplying both sides by $e^{2\pi\iota x}$, integrating between the limits one and zero, and applying the expansion

$$\frac{f(x)}{\theta_{00}(x)} = \frac{1}{\theta_{10}} \sum_{n=\pm\infty} \frac{q^{(n+\frac{1}{2})^2}}{1+q^{2n+1}e^{-2\pi\iota x}} = \frac{1}{2} - \frac{2\iota}{\theta_{10}} \sum_{n=0}^{\infty} \sum_{t=1}^{\infty} (-1)^t q^{(n+\frac{1}{2})^2 + t(2n+1)} \sin 2\pi tx.$$

$$\sum_{t=\pm\infty}\sum_{M=1}^{\infty}(-1)^{M+1}F(M)\,q^{M+3t^2} + \sum_{t=\pm\infty}\sum_{M=1}^{\infty}(-1)^{t+1}F(M)\,q^{M+3t^2}=\sum_{n=0}^{\infty}2n\left[B_{2n}-B_{-2n}\right].$$

Equating coefficients of q^{2m}, we have

$$F(2m)-2F(2m-3.1^2)+2F(2m-3.2^2) - 2F(2m-3.3^2)+\ldots=(-1)^{m+1}\Sigma x$$

where $$x^2-3y^2=m,$$

with $x>0$ and $-\frac{1}{3}(x-1)\leq y\leq x/3$.

§ 6. We can now consider other functions similar to to $f(x)$. Thus, putting

$$F(x)=\Sigma\, q^{n^2+4}e^{n\pi\iota x}/(1-q^n),$$

where the summation for n includes all odd positive and negative integers, we have

$$\begin{aligned} F(x+1)&=-F(x),\\ F(x)-F(x+\omega)&=\theta_{10}(x),\\ F(x+\omega)&=-F(-x),\\ 2F(0)&=\theta_{10},\\ 2F(\tfrac{1}{2})&=\iota\theta_{00}\theta_{10}. \end{aligned}$$

And if n is a positive integer,

$$F(x)-F(x+n\omega)=\theta_{10}(x)\left[1+q^{-1}e^{-2\pi\iota x}+\ldots q^{-(n-1)^2}e^{-2\pi\iota(n-1)x}\right],$$

$$F(x-n\omega)-F(x)=\theta_{10}(x)\left[q^{-1}e^{2\iota\pi x}+\ldots\, q^{-n^2}e^{2n\pi\iota x}\right],$$

This is not useful for class relation formulæ because of the denominator θ_{10}, but the same method gives us the expansion of

$$\frac{f(x)}{\theta_{00}(x)}\,\theta_{00}[(2m+1)x,\ (2m+1)\omega].$$

We may note that if we differentiate the expansion for $f(x)/\theta_{00}(x)$, we find

$$\sum_{n=1}^{\infty}F(n)\,q^n=\frac{1}{\theta_{10}}\sum_{n=0}^{\infty}\frac{q^{(n+\frac{1}{2})^2}}{(1-q^{2n+1})^2},$$

a result due to Kronecker.

$$2F(n\omega) \quad = -\theta_{10}[1+2q^{-1}+\dots 2q^{-(n-1)^2}] = -\theta_{10}S_n, \text{ say,}$$

$$2F(-n\omega) = \quad \theta_{10}[1+2q^{-1}+\dots 2q^{-n^2}] = \theta_{10}S_{n+1},$$

$$F(\tfrac{1}{2}\pm n\omega) = F(\tfrac{1}{2}).$$

Now consider the integral

$$\int \frac{F(t)\cot\pi(t-x)}{\theta_{11}(t)}dt = \int K dt, \text{ say,}$$

taken around the sides of the parallelogram $ABCD$ of § 2. The integrals along DA and BC cancel each other, since the integrand admits unity as a period. Hence, since the singularities of the integrand, within the parallelogram $ABCD$ are at $t=x$,* $0, \pm\omega, \dots, \pm n\omega$, we find

$$\frac{1}{\pi}\frac{F(x)}{\theta_{11}(x)} + \sum_{r=n}^{-n} \frac{(-1)^r q^{r^2} F(r\omega)\cot\pi(r\omega - x)}{\theta_{11}'} = \frac{1}{2\pi\iota}\int_A^B K dt + \frac{1}{2\pi\iota}\int_C^D K dt.$$

To examine the behaviour of the right-hand side, when $n\to\infty$, put $t = n\omega + s$ in the first integral, which is then seen to vanish, and $t = -n\omega + s$ in the second, which becomes

$$\frac{1}{2\pi\iota}\int_{(-1-\omega)/2}^{(1-\omega)/2} \frac{F(s)+\theta_{10}(s)[q^{-1}e^{2\pi\iota s}+\dots q^{-n^2}e^{2n\pi\iota s}]}{(-1)^n q^{-n^2} e^{2n\pi\iota s}\,\theta_{11}(s)} \times \cot\pi(-n\omega+s-x)\,ds,$$

and tends to

$$\frac{(-1)^n}{2\pi}\int_{(-1-\omega)/2}^{(1-\omega)/2} \frac{\theta_{10}(s)}{\theta_{11}(s)}ds$$

when $n\to\infty$, or putting $s = -\frac{1}{2}\omega - \xi$, this is

$$\frac{\iota(-1)^n}{2\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\theta_{00}(\xi)}{\theta_{01}(\xi)}d\xi \text{ or } (-1)^n\,\iota/2\pi\,\theta_{00}\,\theta_{01}$$

on expanding the integrand as a Fourier series, or by considering

$$\int \frac{\theta_{00}(\xi)}{\theta_{01}(\xi)}d\xi$$

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* The real part of x is supposed to be between 0 and 1. The expansion, of course, holds for all values of x.

taken around the parallelogram whose angular points are given by $\pm\frac{1}{2}\pm\omega$.

Substituting now for $F(r\omega)$, and noting $\theta_{11}' = \pi\theta_{00}\theta_{01}\theta_{10}$, we have

$$2\theta_{00}\theta_{01}F(x)/\theta_{11}(x) = \sum_{r=0}^{n}(-1)^r q^{r^2}\cot\pi(r\omega - x) \times[1+2q^{-1}+\dots 2q^{-(r-1)^2}]$$

$$+\sum_{r=1}^{\infty}(-1)^r q^{r^2}\cot\pi(r\omega+x)[1+2q^{-1}+\dots 2q^{-r^2}]+\iota(-1)^n,$$

where $n\to\infty$.

Multiplying by $\theta_{11}(x)$, differentiating and putting $x=0$, the result can be written as

$$2F'(0)/\pi\theta_{10} = \sum_{n=1}^{n}(-1)^n q^{n^2}\cot(n\pi\omega)\,S_n + \sum_{n=1}^{n}(-1)^n q^{n^2}\cot(n\pi\omega)\,S_{n+1}+\iota(-1)^n,$$

where $n\to\infty$.

The right-hand side is

$$2\iota\sum_{n=1}^{n}(-1)^{n+1}q^{n^2}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)(1+2q^{-1}+\dots 2q^{-(n-1)^2}+q^{-n^2})+\iota(-1)^n,$$

hence

$$2F'(0)/\iota\pi\theta_{10} = 2\sum_{n=1}^{n}(-1)^{n+1}q^{n^2}\left(\frac{1+q^{2n}}{1-q^{2n}}\right) \times(1+2q^{-1}+\dots 2q^{-(n-1)^2})+2\sum_{n=1}^{n}(-1)^{n+1}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)+(-1)^n,$$

where $n\to\infty$.

In the first summation put

$$\frac{1+q^{2n}}{1-q^{n}} = 1+2q^{2n}+2q^{4n}+\dots = 2\sum_{t=0}^{\infty}q^{2nt},$$

provided we halve the coefficient when $t=0$; in the second write

$$\frac{1+q^{2n}}{1-q^{2n}} = 1+\frac{2q^{2n}}{1-q^{2n}},$$

and note that

$$2\sum_{n=1}^{n}(-1)^{n+1}+(-1)^n = 1.$$

Hence

$$2F'(0)/\iota\pi\theta_{10} = 2\Sigma(-1)^{n+1}\left[2q^{n^2-r^2+2nt}\right] + 4\Sigma(-1)^{n+1}q^{2nt} + 1,$$

where n takes all positive integer values, excluding zero,

r takes the values $0, \pm 1, \pm 2, \ldots, \pm(n-1)$,

t takes the values $0, 1, 2, \ldots$,

provided that, when $t = 0$, we halve the coefficient of $q^{n^2-r^2+2nt}$, and replace that of q^{2nt} by zero.

This may be rewritten as

$$2F'(0)/\iota\pi\theta_{10} = 1 + 4\Sigma(-1)^{n+1}q^{n^2-r^2+2nt},$$

where n takes all positive integer values,

r takes the values $0, \pm 1, \pm 2, \ldots, \pm(n-1), n$,

t takes the values $0, 1, 2, \ldots$,

provided that, when $t = 0$, we replace the coefficient 4 by 2; but if, in addition, $r = n$ we replace the coefficient 4 by zero.

It will now be shown that this result can be written in the form

$$2F'(0)/\iota\pi\theta_{10} = 4\sum_{M=0}^{\infty}\left[4F(M) - 3G(M)\right]q^M,$$

so that

$$F'(0)/\theta_{10} = 2\iota\pi\sum_{M=0}^{\infty}\left[4F(M) - 3G(M)\right]q^M.$$

We now find the number of solutions of

$$M = n^2 - r^2 + 2nt, \quad M \neq 0,$$

$$n = 1, 2, \ldots,$$

$$r = 0, \pm 1, \pm 2, \ldots, \pm(n-1), n,$$

$$t = 0, 1, 2, \ldots,$$

remembering that we take only half the number of particular solutions for which $t = 0$. The procedure is the same as in § 3. Thus, when n is odd, we consider the following quadratic forms of determinant $-M$,

$(n, r, n + 2t)$, $\quad r = 0, \pm 1, \ldots, \pm\frac{1}{2}(n-1)$,

$(n, n + r, 2n + 2r + 2t)$, $\quad r = -\frac{1}{2}(n+1), \ldots, -(n-1)$,

$(n, -n + r, 2n - 2r + 2t)$, $\quad r = \frac{1}{2}(n+1), \ldots, n$,

or say

$$(n, \quad \xi, \quad n+2t), \quad \xi = 0, \pm 1, \ldots, \pm \tfrac{1}{2}(n-1),$$

$$(n, \quad \xi, \quad 2\xi + 2t), \quad \xi = 1, 2, \ldots, \tfrac{1}{2}(n-1),$$

$$(n, -\xi, 2\xi + 2t), \quad \xi = 0, 1, \ldots, \tfrac{1}{2}(n-1).$$

The number of solutions corresponding to the values $r = 0, \pm 1, \ldots, \pm \frac{1}{2}(n-1)$ is the sum of what we have called $(0, 0)$ and the number of ambiguous forms of the type $(2a+1, 0, 2a+1+2t)$, noting that we take only half the number of the forms $(2a+1, 0, 2a+1)$. For the remaining values of r the number of solutions is $(0, e) + (e, 0)$ plus the number of ambiguous forms of the types $(2a+1, 0, 2b)$, $(n, \xi, 2\xi)$, $\xi = 1, 2, \ldots, \frac{1}{2}(n-1)$. Reckoning the form $(2a+1\ 0, 2a+1)$ as $\frac{1}{2}$, the total is evidently $F(M)$.

When n is even, we have the scheme

$$(n, \quad \xi, \quad n+2t), \ \xi = 0, \pm 1, \ldots \pm \tfrac{1}{2}n, \qquad \xi = r,$$

$$(n, \quad \xi, \ 2\xi + 2t), \ \xi = 1, 2, \ldots \qquad \tfrac{1}{2}n - 1, \ \xi = n + r,$$

$$(n, -\xi, 2\xi + 2t), \ \xi = 0, 1, 2, \ldots \quad \tfrac{1}{2}n - 1, \ \xi = n - r.$$

Excluding for a moment the values of r, t corresponding to ambiguous forms, the number of solutions when $r = 0, \pm 1, \ldots, \pm \frac{1}{2}n$ is equal to the number of the even, non-ambiguous, reduced forms of determinant $-M$; for the remaining values of r we must double this number. The number of solutions corresponding to the ambiguous forms requires a little care. Thus, when $\xi = 0$, the number of solutions is equal to the number of forms of the types $(n, 0, n+2t)$ and $(n, 0, 2t)$, and this is three times the number of reduced forms of the type $(2a, 0, 2b)$, since $(n, 0, n)$ is reckoned as $\frac{1}{3}$. For $t = 0$ in the last two lines of the scheme, and $\xi = \pm \frac{1}{2}n$ in the first line, the number of solutions is equal to the number of forms of the types $(n, \xi, 2\xi)$, $\xi = 1, 2, \ldots, \frac{1}{2}n - 1$, and $[n, \pm \frac{1}{2}n, n+2t]$; that is, three times the number of reduced forms of the type $\{2a, a, b\}$, since when $t = 0$, we take only one of the forms $[n, \pm \frac{1}{2}n, n]$ and reckon it as $\frac{1}{3}$.

When $t = 0$ in the first line of the scheme (omitting $\xi = 0, \pm \frac{1}{2}n$, which have already been considered), or $\xi = 1$, $t = \frac{1}{2}n - 1$; $\xi = 2$, $t = \frac{1}{2}n - 2$, etc., in the last two lines, we have three times the number of reduced forms of the type

$(2a,\ b,\ 2a)$ $b = 1, 2, \ldots, a-1$. Hence the total number of solutions when n is even is equal to three times the number of even forms of determinant $-M$, that is $3\{G(M) - F(M)\}$. Hence the coefficient of q^M is $F(M) - 3\{G(M) - F(M)\}$ or $4F(M) - 3G(M)$.

We can now find an expansion for

$$\Phi(x) = \frac{F(x)}{\theta_{10}(x)}\,\theta_{g,h}(x+\xi)$$

if we carry out the process in § 4. For

$$\Phi(x+1) = (-1)^g\,\Phi(x),$$

and

$$\Phi(x) - (-1)^h e^{2\pi\iota\xi}\,\Phi(x+\omega) = \theta_{g,h}(x+\xi) = \sum_{n=\infty}^{-\infty} A_n e^{n\pi\iota(x+\xi)},$$

and hence

$$\Phi(x) = \sum_{n=\infty}^{-\infty} \frac{A_n e^{n\pi\iota(x+\xi)}}{1 - q^n(-1)^h e^{2\pi\iota\xi}} + \frac{\iota\theta_{00}\theta_{10}}{2}\,\frac{\theta_{g+1,h}(x+\xi)}{\theta_{1,0}(x)}\,\frac{\theta_{g,h}(\frac{1}{2}+\xi)}{\theta_{g+1,h}(\frac{1}{2}+\xi)},$$

since $F(\frac{1}{2}) = \frac{1}{2}\iota\theta_{00}\theta_{10}$.

In particular, if $g = h = 0$,

$$\frac{F(x)\,\theta_{00}(x+\xi)}{\theta_{10}(x)} = \sum_{n=\infty}^{-\infty} \frac{q^{n^2} e^{2n\pi\iota(x+\xi)}}{1 - q^{2n} e^{2\pi\iota\xi}} - \frac{\iota\theta_{00}\theta_{10}}{2}\,\frac{\theta_{10}(x+\xi)}{\theta_{10}(x)}\,\frac{\theta_{01}(\xi)}{\theta_{11}(\xi)}.$$

Differentiating and putting $x = 0$,

$$\frac{F(0)}{\theta_{10}}\,\theta_{00}'(\xi) + \frac{F'(0)}{\theta_{10}}\,\theta_{00}(\xi) = \sum_{n=\infty}^{-\infty} \frac{2n\pi\iota q^{n^2} e^{2n\pi\iota\xi}}{1 - q^{2n} e^{2\pi\iota\xi}} - \frac{\iota\theta_{00}}{2}\,\frac{\theta_{01}(\xi)}{\theta_{11}(\xi)}\,\theta_{10}'(\xi).$$

Adding this equation to the one derived from it by writing $-\xi$ for ξ, we have

$$4\theta_{00}(\xi) \sum_{0}^{\infty} [4F(n) - 3G(n)]\,q^n = \sum_{n=\pm\infty} 2nq^{n^2} e^{2n\pi\iota\xi} \left(\frac{1 + q^{2n} e^{2\pi\iota\xi}}{1 - q^{2n} e^{2\pi\iota\xi}}\right) - \frac{\theta_{00}}{\pi}\,\frac{\theta_{01}(\xi)}{\theta_{11}(\xi)}\,\theta_{10}'(\xi).$$

Putting $\xi = \frac{1}{2}$, we find

$$4\theta_{01} \sum_{0}^{\infty} [4F(n) - 3G(n)]\,q^n = 4 \sum_{n=0}^{\infty} (-1)^n nq^{n^2} \left(\frac{1 - q^{2n}}{1 + q^{2n}}\right) + \theta_{00}^{\,3}\theta_{01},$$

and adding this to the equation derived from equation (j) by putting $\xi = \frac{1}{2}$, we find the classical expansion

$$\theta_{00}^{3} = \sum_{n=0}^{\infty} [24F(n) - 12G(n)]q^n.$$

Putting now $\xi = 0$,

$$4\theta_{00} \sum_{n=0}^{\infty} [4F(n) - 3G(n)]q^n = \sum_{n=1}^{\infty} 4nq^{n^2}\left(\frac{1+q^{2n}}{1-q^{2n}}\right) - \frac{1}{\pi}\frac{\theta_{01}\theta_{10}''\theta_{00}}{\theta_{11}'}.$$

But

$$-\frac{1}{\pi}\frac{\theta_{01}\theta_{10}''\theta_{00}}{\theta_{11}'} = -\frac{1}{\pi^2}\frac{\theta_{10}''}{\theta_{10}} = -\frac{1}{\pi^2}\frac{d}{dx}\left[\frac{\theta_{10}'(x)}{\theta_{10}(x)}\right] \text{ when } x = 0$$

$$= 1 + 8\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2},$$

so that equating coefficients of q^m

$$8\sum_{r} F(m-r^2) - 6\sum_{r} G(m-r^2) = 4\Sigma a + 4\Sigma(-1)^{c+1}c,$$

where the summation for r extends to all positive, negative, and zero values for which $m - r^2$ is not negative, and the summation for c extends to all divisors of m, whose conjugate divisors are even. Since

$$8\sum_{r} F(m-r^2) = -8\Sigma a + 8\Sigma b,$$

by subtraction

$$3[G(m) + 2G(m-1^2) + 2G(m-2^2) + \ldots]$$

$$= -6\Sigma a + 4\Sigma b + 2\Sigma(-1)^c c,$$

a result given by Kronecker in the case when m is odd.

ON THE MOTION OF CONTINUOUS DISTRIBUTIONS OF ELECTRICITY.

By *Dr. H. Bateman.*

§ 1. If we take it for granted that light and electromagnetic disturbances are propagated either by or through a medium of some sort, the question naturally arises whether the substance of which this medium is composed is essentially different from the substance which is commonly associated with the elementary electrified particles or electrons. In endeavouring to answer this question our only way at present is to examine the consequences of certain hypotheses and the simplest hypothesis we can start with is that there is one primeval substance from which both matter and æther are built up.

Starting with the fundamental equations of the theory of electrons

$$\left.\begin{aligned} &\operatorname{rot}\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{\rho\mathbf{v}}{c}, \quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \\ &\operatorname{div}\mathbf{E}=\rho, \qquad\qquad \operatorname{div}\mathbf{H}=0 \end{aligned}\right\} \quad \ldots\ldots(1),$$

we may consider the hypothesis that in the æther ρ is vanishingly small, but $\mathbf{v}$ is finite and satisfies the conditions which according to E. Cunningham* the velocity of the æther must satisfy. Within the electrified particles we may assume as usual that both ρ and $\mathbf{v}$ are finite and that v is numerically less than c.

Let us first of all examine the case when ρ is not actually zero in the æther.† For the sake of simplicity we shall consider the special electromagnetic fields in which the scalar

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* *Proc. Roy. Soc.*, vol. lxxxiii. (1909), p. 110. *The Principle of Relativity*, chap. xv.

† It may be desirable to divide an arbitrary field into component fields, one of which is associated with the positive charges and the other with the negative; ρ may then be zero for part of the total field and not zero for the component fields. The resolution into positive and negative fields has been considered by H. A. Lorentz, *Amsterdam Proc.*, t. 8 (1900), p. 603. R. Gans, *Jahresbericht d. Deutsch. Math. Verein.* Bd. 14 (1905), p. 578. J. J. Thomson, *Proc. Camb. Phil. Soc.* (1909), p. 65. D. L. Webster, *Proc. Amer. Acad. of Arts-Sciences* (1912), p. 509.

product (EH) is everywhere zero. Cunningham's condition then takes one of the two forms*

$$\mathrm{H} - \frac{1}{c}[\mathrm{vE}] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\mathrm{E} + \frac{1}{c}[\mathrm{vH}] = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Electromagnetic fields in which conditions (1) and (2) are satisfied have been discussed by Sir Joseph Thomson in his theory of moving Faraday tubes. A case of particular interest occurs when the components of E and H are given by expressions of type

$$E_x = \psi \frac{\partial(\theta, \phi)}{\partial(y, z)}, \quad H_x = -\frac{1}{c}\psi \frac{\partial(\theta, \phi)}{\partial(x, t)} \quad \ldots\ldots\ldots (4),$$

for then a moving line of electric force is given by the equations $\theta = \text{const.}$, $\phi = \text{const.}$, and since we have also

$$\rho v_x = -\frac{\partial(\theta, \phi, \psi)}{\partial(y, z, t)}, \quad \rho = \frac{\partial(\theta, \phi, \psi)}{\partial(x, y, z)}, \text{ etc.},$$

it follows that a particle of electricity moves along a line for which θ, ϕ, and ψ are constant. *Hence in this case a moving line of electric force is always composed of the same particles of electricity.*

If, as we are entitled to do† in virtue of the equations

$$\mathrm{rot}\,\mathrm{E} = -\frac{1}{c}\frac{\partial \mathrm{H}}{\partial t}, \quad \mathrm{div}\,\mathrm{H} = 0, \quad (\mathrm{EH}) = 0,$$

we write the components of E and H in *Hargreaves' form*‡

$$H_x = \frac{\partial(\alpha, \beta)}{\partial(y, z)}, \quad E_x = \frac{1}{c}\frac{\partial(\alpha, \beta)}{\partial(x, t)} \quad \ldots\ldots\ldots (5),$$

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* This may be seen at once from equations (15), (16), (17) of my paper on "Time and Electromagnetism," *Messenger of Mathematics*, vol. xlv., pp. 97–115, or from the geometrical construction in which the possible velocities of type V are represented by the points on two lines which are polar lines with respect to a sphere S. Since only one of these two lines can meet the sphere S in real points follows that if a velocity v less than c can be found for which (2) is satisfied, there is no velocity w less than c for which $\mathrm{E} + \frac{1}{c}[\mathrm{wH}] = 0$.

† Cf. Méray, *Annales de l'école normale* (1899). The theorem may be deduced from the fact that if A and Φ are the potentials from which E and H are derived, the expression $A_x dx + A_y dy + A_z dz - c\Phi dt$ can be reduced to the form $d\gamma + \alpha d\beta$ when $(\mathrm{EH}) = 0$. Cf. Forsyth's *Differential Equations* (1914), Ex. 4, p. 338.

‡ R. Hargreaves, *Proc. Camb. Phil. Soc.* (1915).

we can prove by an argument similar to that given on p. 113 of my former paper* that

$$dx^2+dy^2+dz^2-c^2dt=Ad\alpha^2+2Cd\alpha d\beta+Bd\beta^2$$
$$+Pd\theta^2+2Rd\theta d\phi+Qd\phi^2\ldots\ldots(6),$$

where
$$PQ-R^2=-\psi^2(AB-C^2)\ldots\ldots\ldots\ldots(7).$$

In order to determine an electromagnetic field in which the components of the vectors **E** and **H** are specified by expressions of types (4) and (5), it is only necessary to find a transformation from (x, y, z, t) to $(\alpha, \beta, \theta, \phi)$, which leads to a relation of type (6). The function (6) can then be derived from formula (7). To verify that equation (2) is indeed satisfied, we note that

$$\rho v_x=+\frac{\partial(\theta,\phi,\psi,x)}{\partial(x,y,z,t)}=+\frac{\partial(\theta,\phi,\psi,\chi)}{\partial(x,y,z,t)}\frac{\partial x}{\partial\chi}=+J\frac{\partial x}{\partial\chi},$$

where χ is some function independent of θ, ϕ, ψ. Similarly we find from (4) that

$$E_x=\psi J\frac{\partial(x,t)}{\partial(\psi,\chi)},\quad H_x=-\frac{1}{c}\psi J\frac{\partial(y,z)}{\partial(\psi,\chi)}.$$

Since $\rho=J\dfrac{\partial t}{\partial\chi}$, it is easy to see that (2) is satisfied. Equation (2) is of course a particular case of the condition† that the changes in an electromagnetic field can be supposed to be produced by the convection of tubes of electric force. It follows from the remark in the third foot-note that when the velocity of an arbitrary point on a moving line of electric force is less than that of light, the velocity of an arbitrary point on a moving line of magnetic force is greater than that of light, provided of course that **E** and **H** are perpendicular as we have supposed. Both velocities can, however, be equal to the velocity of light in the special case when **E** and **H** are equal in magnitude.

A particular solution with a point singularity S moving

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* *Loc. cit.* There should be a minus sign in front of λ^2 in equation (25) and in the last equation but one.

† J. Larmor, International Congress of Mathematics, *Proceedings*, vol. i., p. 210 (1912).

with a velocity w, which varies but always remains less in magnitude than the velocity of light, is obtained by writing

$$\alpha = t - \frac{r}{c}, \quad \beta = \frac{(\sigma \mathrm{n}) + \mu}{cr - (\mathrm{wn})},$$

where r denotes the distance of the point x, y, z from the position of S at time α, n is the radius vector from the second point to the first, the vector σ and the scalar μ are arbitrary functions of α. This solution, which includes Liénard's solution for a point charge in 'free æther' as a particular case, will be discussed in a later paper. In general electricity is fired out in all directions from the different positions of the singularity S, and travels along straight lines with the velocity of light, the electric charge associated with the singularity S generally varies, but if μ is a constant multiple of $c^2 - w^2$, the charge remains constant.

§ 2. If we consider a second electromagnetic field for which*

$$E_x^0 = \psi_0 \frac{\partial(\alpha_0, \phi_0)}{\partial(y, z)} = \frac{1}{c} \frac{\partial(\alpha_0, \beta_0)}{\partial(x, t)},$$

$$H_x^0 = -\frac{1}{c} \psi_0 \frac{\partial(\theta_0, \phi_0)}{\partial(x, t)} = \frac{\partial(\alpha_0, \beta_0)}{\partial(y, z)},$$

etc., the condition $(\mathrm{EH}^0) = (\mathrm{E}^0\mathrm{H}) = 0$ will be satisfied if

$$\frac{\partial(\theta, \phi, \theta_0, \phi_0)}{\partial(x, y, z, t)} = 0 \text{ and } \frac{\partial(\alpha, \beta, \alpha_0, \beta_0)}{\partial(x, y, z, t)} = 0,$$

that is, if there is a relation between $\theta, \phi, \theta_0, \phi_0$ and also a relation between $\alpha, \beta, \alpha_0, \beta_0$. This means that a moving line of electric force $\theta_0 = \text{const.}$, $\phi_0 = \text{const.}$, which once intersects the moving line of electric force $\theta = \text{const.}$, $\phi = \text{const.}$, will continue to intersect it, and similarly that there is a continued incidence of lines of magnetic force belonging to the two fields.† The lines of force of the two fields thus form a kind of network. The interpretation of the condition

$$(\mathrm{EE}^0) - (\mathrm{HH}^0) = 0$$

is somewhat similar.

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* In particular if $\theta_0 = \theta$, $\phi_0 = \psi$, $\psi_0 = \phi$, the condition $(EH^0) + (E^0H) = 0$ is satisfied, and when the two fields are superposed the total volume density of electricity is zero.

† It should be noticed, however, that if two lines of force intersect in more than one point, two of the points of intersection can approach one another, coincide, and then become imaginary.

In this case we have

$$\frac{\partial(\theta,\ \phi,\ \alpha_0,\ \beta_0)}{\partial(x,\ y,\ z,\ t)} = 0, \quad \frac{\partial(\theta_0,\ \phi_0,\ \alpha,\ \beta)}{\partial(x,\ y,\ z,\ t)} = 0,$$

and these equations imply that there is continued incidence of a line of electric force belonging to one field with a line of magnetic force belonging to the other. The conditions $(\mathbf{E}\mathbf{H}^0) + (\mathbf{E}^0\mathbf{H}) = 0$, $(\mathbf{E}\mathbf{E}^0) - (\mathbf{H}\mathbf{H}^0) = 0$, are both satisfied in the case of conjugate fields.

§ 3. Electromagnetic fields for which equations (1) and (3) are satisfied have been discussed by Levi Civita,* and have been called by him *pure electromagnetic fields*. In this case the total force exerted by the field on each particle of electricity is null, consequently if the equations of motion of a finite quantity of electricity are supposed to imply that the integral of the total force for each discrete mass is zero, as in the theory of Lorentz, then the equations of motion will be satisfied because each element of the integral will be zero. Electromagnetic fields which possess this property are thus of considerable interest, a few types have already been found by Italian writers.† The condition (3) is generally taken in the form

$$\rho\mathbf{E} + \frac{1}{c}[\rho\mathbf{v}.\mathbf{H}] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3a),$$

which is satisfied also when $\rho = 0$ and $\rho\mathbf{v} = 0$.

If we assume that the components of $\mathbf{E}$ and $\mathbf{H}$ can be expressed in Hargreaves' form (5), the formula

$$E_x + \frac{v_y}{c} H_z - \frac{v_z}{c} H_y = \frac{1}{c}\left(\frac{\partial\alpha}{\partial x}\frac{d\beta}{dt} - \frac{\partial\beta}{\partial x}\frac{d\alpha}{dt}\right) \ldots\ldots (8)$$

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z} \ldots\ldots\ldots\ldots (9),$$

which has been given by Hargreaves,‡ indicates that when (3) is satisfied we must have $\frac{d\alpha}{dt} = 0$, $\frac{d\beta}{dt} = 0$, otherwise $\mathbf{H}$

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* T. Levi Civita, "Sur le mouvement de l'électricité sans liaisons ni forces extérieures," *Comptes Rendus*, t. 145 (1907), p. 417. *Ven. Ist. Atti*, t. 67, p. 995 *Rend. Lincei*. (1909), 1° Sen., p 83.

† L. Caffaratti, *Nuovo Cimento* (5), t. 15, p. 369, t. 16, p. 376. V. Cisotti, *Ven. Ist. Atti*, t. 68, p. 361. *Nuovo Cimento* (5), t. 17, p. 103.

‡ *Loc. cit.*

would be zero. These equations imply that α and β are constant for each particle of electricity. *Consequently a moving line of magnetic force whose equations are* $\alpha = const.$, $\beta = const.$, *always consists of the same particles of electricity.*

The differential equations which α and β must satisfy in order that the field (5) may be pure are easily found by direct substitution of the values of ρ and $\rho\mathbf{v}$ in the equations

$$\rho \frac{d\alpha}{dt} = 0, \quad \rho \frac{d\beta}{dt} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (10).$$

The first equation is

$$\left(\frac{\partial\alpha}{\partial x}\right)^2 \frac{\partial^2\beta}{\partial x^2} + \left(\frac{\partial\alpha}{\partial y}\right)^2 \frac{\partial^2\beta}{\partial y^2} + \left(\frac{\partial\alpha}{\partial z}\right)^2 \frac{\partial^2\beta}{\partial z^2} + \frac{1}{c^4}\left(\frac{\partial\alpha}{\partial t}\right)^2 \frac{\partial^2\beta}{\partial t^2} + 2\frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial z}\frac{\partial^2\beta}{\partial y\partial z}$$

$$+ 2\frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial x}\frac{\partial^2\beta}{\partial z\partial x} + 2\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y}\frac{\partial^2\beta}{\partial x\partial y} - \frac{2}{c^2}\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial t}\frac{\partial^2\beta}{\partial x\partial t} - \frac{2}{c^2}\frac{\partial\alpha}{\partial y}\frac{\partial\alpha}{\partial t}\frac{\partial^2\beta}{\partial y\partial t}$$

$$- \frac{2}{c^2}\frac{\partial\alpha}{\partial z}\frac{\partial\alpha}{\partial t}\frac{\partial^2\beta}{\partial z\partial t} - \left[\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\alpha}{\partial t}\right)^2\right]$$

$$\times \left[\frac{\partial^2\beta}{\partial x^2} + \frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\beta}{\partial t^2}\right] = \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x}\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y}\frac{\partial^2\alpha}{\partial y^2}$$

$$+ \frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z}\frac{\partial^2\alpha}{\partial z^2} + \frac{1}{c^2}\frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial t}\frac{\partial^2\alpha}{\partial t^2} + \left(\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial z} + \frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial y}\right)\frac{\partial^2\alpha}{\partial y\partial z}$$

$$+ \left(\frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial z}\right)\frac{\partial^2\alpha}{\partial z\partial x} + \left(\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial x}\right)\frac{\partial^2\alpha}{\partial x\partial y}$$

$$- \frac{1}{c^2}\left(\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial t} + \frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial x}\right)\frac{\partial^2\alpha}{\partial x\partial t} - \frac{1}{c^2}\left(\frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial t} + \frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial y}\right)\frac{\partial^2\alpha}{\partial y\partial t}$$

$$- \frac{1}{c^2}\left(\frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial t} + \frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial z}\right)\frac{\partial^2\alpha}{\partial z\partial t} - \left[\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial y}\frac{\partial\beta}{\partial y} + \frac{\partial\alpha}{\partial z}\frac{\partial\beta}{\partial z} - \frac{1}{c^2}\frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial t}\right]$$

$$\times \left[\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} + \frac{\partial^2\alpha}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2\alpha}{\partial t^2}\right] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The second equation is obtained from this by interchanging α and β.

In particular, if $\beta = z$, the differential equations for α are

$$\frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2\alpha}{\partial t^2}, \quad \frac{\partial}{\partial z}\left[\left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 - \frac{1}{c^2}\left(\frac{\partial\alpha}{\partial t}\right)^2\right] = 0 \ldots. (12).$$

In this case $E_x = -\frac{1}{c}\frac{\partial \alpha}{\partial t}$, $E_y = 0$, $E_z = 0$, $H_x = \frac{\partial \alpha}{\partial y}$, $H_y = -\frac{\partial \alpha}{\partial x}$, $H_z = 0$, hence

$$\rho = -\frac{1}{c}\frac{\partial^2 \alpha}{\partial z \partial t}, \quad \rho v_x = c\frac{\partial^2 \alpha}{\partial x \partial z}, \quad \rho v_y = c\frac{\partial^2 \alpha}{\partial y \partial z}, \quad \rho v_z = 0.$$

This means that the electricity flows in a system of parallel planes. A particular solution is obtained by writing

$$\alpha = zf(x+ct) + \frac{1}{z}g(x-ct) \ldots\ldots\ldots\ldots (13).$$

In this case $v_y = 0$, while

$$v_x = c\frac{g'(x-ct) - z^2 f''(x+ct)}{g'(x-ct) + z^2 f''(x+ct)}.$$

If $f''(x+ct)\,g'(x-ct)$ is always positive, the velocity v_x is less than the velocity of light. Since ρ is infinite when $z = 0$ this plane must be excluded. It seems reasonable to suppose that the motion takes place between two perfectly conducting planes $z = h_1$ and $z = h_2$, where h_1 and h_2 are both positive. The functions f' and g' should perhaps in this case be supposed to vanish for large values of their arguments.

If $\beta = t$ the differential equations for α are*

$$\frac{\partial^2 \alpha}{\partial x^2} + \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial^2 \alpha}{\partial z^2} = 0, \quad \frac{\partial}{\partial t}\left[\left(\frac{\partial \alpha}{\partial x}\right)^2 + \left(\frac{\partial \alpha}{\partial y}\right)^2 + \left(\frac{\partial \alpha}{\partial z}\right)^2\right] = 0 \ldots (14).$$

A particular solution is obtained by writing

$$\alpha = e^{ikt} f(\omega) + e^{-ikt} f(\varpi) \ldots\ldots\ldots\ldots (15),$$

where ω and ϖ are two conjugate complex quantities satisfying the equation

$$\left.\begin{aligned} a\omega &= xl(\omega) + ym(\omega) + zn(\omega) \\ l^2 + m^2 + n^2 &= 0 \end{aligned}\right\} \ldots\ldots\ldots (16),$$

where l, m, and n are functions of ω, and a and k are arbitrary constants.† In this case we have an electric field which changes periodically, but there is no magnetic field; the volume density of electricity is zero, but there are convection currents. The electric force is at right angles to the electric

* In the present case, since $\rho = 0$ and $H = 0$, the condition (3*a*) is satisfied in virtue of the first equation alone: to obtain the second equation we must assume that in addition to (3*a*) the relation $(\rho v.E) = 0$ is also satisfied.

† Jacobi has shown that $f(\omega)$ and $f(\varpi)$ are solutions of Laplace's equation. See the author's *Electrical and Optical Wave Motion*, pp. 114, 135.

current, and does not alter in magnitude with the time, although its direction may be continually changing. In the case when

$$\omega = \frac{x+iy}{z+r}, \quad \varpi = \frac{x-iy}{z+r}, \quad r^2 = x^2+y^2+z^2\ldots,$$

the electric force and convection current are both perpendicular to the radius from the origin. To avoid places where the electric force is infinite, we may imagine the field to be bounded by surfaces built up of lines of flow of convection currents.

It should be noticed that in the present case H is zero, consequently we cannot conclude that α and β are constant during the motion of the electricity.

Let us next consider the case when $\beta = T$, and there is a relation of the form

$$dx^2+dy^2+dz^2-c^2dt^2 = A^2dX^2+B^2dY^2+C^2dZ^2-D^2dT^2\ldots(17).$$

The integral form $d(\alpha, \beta)$ may now be written in the form

$$\frac{\partial\alpha}{\partial X}d(X, T)+\frac{\partial\alpha}{\partial Y}d(Y, T)+\frac{\partial\alpha}{\partial Z}d(Z, T)\ldots.(18),$$

and the reciprocal form* is

$$\frac{BC}{AD}\frac{\partial\alpha}{\partial X}d(Y, Z)+\frac{CA}{BD}\frac{\partial\alpha}{\partial Y}d(Z, X)+\frac{AB}{CD}\frac{\partial\alpha}{\partial Z}d(X, Y)\ldots(19).$$

Operating on this with

$$\frac{\partial}{\partial X}dX+\frac{\partial}{\partial Y}dY+\frac{\partial}{\partial Z}dZ+\frac{\partial}{\partial T}dT,$$

we get

$$\left[\frac{\partial}{\partial X}\left(\frac{BC}{AD}\frac{\partial\alpha}{\partial X}\right)+\frac{\partial}{\partial Y}\left(\frac{CA}{BD}\frac{\partial\alpha}{\partial Y}\right)+\frac{\partial}{\partial Z}\left(\frac{AB}{CD}\frac{\partial\alpha}{\partial Z}\right)\right]d(X, Y, Z)$$

$$+\frac{\partial}{\partial T}\left(\frac{BC}{AD}\frac{\partial\alpha}{\partial X}\right)d(Y, Z, T)+\frac{\partial}{\partial T}\left(\frac{CA}{BD}\frac{\partial\alpha}{\partial Y}\right)d(Z, X, T)$$

$$+\frac{\partial}{\partial T}\left(\frac{AB}{CD}\frac{\partial\alpha}{\partial Z}\right)d(X, Y, T)\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

* For the method used here we may refer to *Proc. London Math. Soc.* (2), vol. viii. (1910), p. 223. Ricci and Levi Civita, *Math. Ann.* Bd. 54 (1901). Th. de Donder, *Rend. Palermo* (1901). The method of multiplication of integral forms is due I believe to E. Cartan, *Annales de l'ecole normale* (1899).

Since (18) is the same as

$$H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + cE_x d(x, t) + cE_y d(y, t) + cE_z d(z, t) \ldots\ldots\ldots (21),$$

(19) is a constant multiple of

$$E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - cH_x d(x, t) - cH_y d(y, t) - cH_z d(z, t),$$

and (20) is a constant multiple of

$$\rho d(x, y, z) - \rho v_x d(y, z, t) - \rho v_y d(z, x, t) - \rho v_z d(x, y, t) \ldots (22).$$

The product* of (22) and (21) gives an integral form of type

$$\rho\, d(x, y, z, t)\,[(cE_x + v_y H_z - v_z H_y)\,\delta x + (cE_y + v_z H_x - v_x H_z)\,\delta y + (cE_z + v_x H_y - v_y H_x)\,\delta z - c(v_x E_x + v_y E_y + v_z E_z)\,\delta t],$$

and this is identically zero on account of $(3a)$ and the additional equation $\rho(\mathbf{vE}) = 0$, which is generally a consequence of $(3a)$, but may not be when $\rho = 0$ and $\rho\mathbf{v} \neq 0$. It follows then that the product of (18) and (20) must vanish identically. This leads to the equations

$$\left.\begin{aligned} \frac{\partial}{\partial X}\left(\frac{BC}{AD}\frac{\partial \alpha}{\partial X}\right) + \frac{\partial}{\partial Y}\left(\frac{CA}{BD}\frac{\partial \alpha}{\partial Y}\right) + \frac{\partial}{\partial Z}\left(\frac{AB}{CD}\frac{\partial \alpha}{\partial Z}\right) &= 0 \\ \frac{\partial \alpha}{\partial X}\frac{\partial}{\partial T}\left(\frac{BC}{AD}\frac{\partial \alpha}{\partial X}\right) + \frac{\partial \alpha}{\partial Y}\frac{\partial}{\partial T}\left(\frac{CA}{BD}\frac{\partial \alpha}{\partial Y}\right) + \frac{\partial \alpha}{\partial Z}\frac{\partial}{\partial T}\left(\frac{AB}{CD}\frac{\partial \alpha}{\partial Z}\right) &= 0 \end{aligned}\right\} (23).$$

If, in particular, we write $x = \rho\cos\phi$, $y = \rho\sin\phi$, we have

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = d\phi^2 + \rho^2 d\phi^2 + dz^2 - c^2 dt^2;$$

hence if $T = \rho$, $X = \phi$, $Y = z$, $Z = t$, we have $A^2 = \rho^2$, $B^2 = 1$, $C^2 = -c^2$, $D^2 = -1$, and the differential equations become

$$\frac{\partial^2 \alpha}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2 \alpha}{\partial \phi^2} - \frac{1}{c^2}\frac{\partial^2 \alpha}{\partial t^2} = 0,$$

$$\frac{\partial \alpha}{\partial z}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \alpha}{\partial z}\right) + \frac{\partial \alpha}{\partial \phi}\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial \alpha}{\partial \phi}\right) - \frac{1}{c^2}\frac{\partial \alpha}{\partial t}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \alpha}{\partial t}\right) = 0.$$

A particular solution is given by

$$\alpha = F(\rho) f(z + ct) + \frac{1}{\rho^2 F(\rho)}\, g(z - ct).$$

* To form the product we replace $d(x, y, z)$ by $\delta x\, d(y, z) + \delta y\, d(z, x) + \delta z\, d(x, y)$ and $d(y, z, t)$, $d(z, x, t)$, $d(x, y, t)$ by similar expressions.

A line of magnetic force $\rho = \text{const.}$, $\alpha = \text{const.}$ is in this case a circle whose radius remains constant. The line of force generally moves, but in special cases it may remain stationary. The electric force at any point is along a line which meets the axis of z at right angles.

§ 4. If we think of a tube of electric force as consisting of a chain of electric doublets, in which the positive and negative charges are practically coincident, it is easy to see that more than one type of disturbance can be propagated along the tube. In the first place the tube can move and change in form without producing any changes in the doublets; for instance, a kink may run along the tube as in Sir Joseph Thomson's theory of the Röntgen rays. Secondly, we may suppose that a doublet at one end of the tube has its moment changed in some way, that this affects the next doublet so as to produce a change in its moment, and so on all along the line. The two charges in a doublet may be supposed to become separated in any direction not necessarily in the direction of the tube. Denoting the two types of disturbance by the terms *kink* and *stretch*, we have apparently two distinct types of electromagnetic radiation. The first type may perhaps be associated with the Liénard-Wiechert solution of Maxwell's equations, which corresponds to a point charge moving in an arbitrary manner with a velocity less than c, while the second type may be associated with the solutions of Maxwell's equations which correspond to the projection of singularities with velocity c from a moving point singularity. These solutions have been discussed on several previous occasions.* In both cases the doublets must be supposed to travel along straight lines with the velocity of light, and this determines the motion of a line of electric force. Radiation seems to be associated with a variation in the direction of projection of doublets belonging to a chain from a moving point charge, and it is easy to see how a kink in a line of force can arise in this way. This is in fact Sir Joseph Thomson's theory of radiation. In the type of field mentioned in § 1 electric charges as well as doublets must be supposed to be fired out from the moving point charge, and again radiation seems to be determined by a variation in the direction of projection, and not necessarily by a change in velocity of the moving point charge.

* *Electrical and Optical Wave Motion*, chap. viii., and *Messenger of Mathematics*, vol. xlvi., pp. 1—11, 1915.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLV.

On a point in the theory of Fourier series.

1. In this note a_0, a_1, a_2, ..., b_1, b_2, ... are the Fourier constants of a summable function $f(x)$, and

$$s_n = \tfrac{1}{2}A_0 + \sum_1^n A_\nu = \tfrac{1}{2}a_0 + \sum_1^n (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

is the sum of the first $n+1$ terms of its Fourier series. I proved in 1913* that

$$s_n = o(\log n), \tag{1.1}$$

and that the series†

$$\Sigma \frac{A_n}{\log n}$$

is convergent, for almost all values of x, and in particular those which satisfy an important criterion of Lebesgue. This criterion is obtained by writing

$$\phi(\alpha) = f(x+\alpha) + f(x-\alpha) - 2f(x) \tag{1.2}$$

and by supposing that

$$\Phi(\alpha) = \int_0^\alpha |\phi(t)|\, dt = o(\alpha) \tag{1.3}$$

when $\alpha \to 0$. It is often expressed by saying that $|\phi(\alpha)|$ *is the differential coefficient of its indefinite integral for* $\alpha = 0$.

2. I shall now prove that the condition (1.3) may be replaced by a slightly more general condition.

Theorem A. *If* $0 < \alpha < \delta$ *and*

$$\Psi(\alpha) = \int_\alpha^\delta |\phi(t)| \frac{dt}{t} = o\left(\log \frac{1}{\alpha}\right) \tag{2.1}$$

when $\alpha \to 0$, *then* $s_n = o(\log n)$.

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* G. H. Hardy, 'On the summability of Fourier's Series', *Proc. London Math. Soc.*, ser. 2, vol. xii. (1913), pp. 365–372.

† The series must obviously be supposed to begin with the term for which $n = 2$.

We have to prove that

$$j_n = \int_0^\delta \phi(t) \frac{\sin nt}{t}\, dt = o(\log n). \tag{2.2}$$

Now

$$\begin{aligned} j_n &= \int_0^{1/n} \phi(t) \frac{\sin nt}{t}\, dt + \int_{1/n}^\delta \phi(t) \frac{\sin nt}{t}\, dt \\ &= O\left\{n \int_0^{1/n} |\phi(t)|\, dt\right\} + O\left\{\int_{1/n}^\delta |\phi(t)| \frac{dt}{t}\right\} \\ &= O\left\{n \Phi\left(\frac{1}{n}\right)\right\} + O\left\{\Psi\left(\frac{1}{n}\right)\right\}. \end{aligned} \tag{2.3}$$

But

$$\begin{aligned} \Phi(\alpha) &= -\int_0^\alpha t \Psi'(t)\, dt = -\alpha \Psi(\alpha) + \int_0^\alpha \Psi(t)\, dt \\ &< \int_0^\alpha \Psi(t)\, dt = \int_0^\alpha o\left(\log \frac{1}{t}\right) dt \\ &= o\left(\alpha \log \frac{1}{\alpha}\right). \end{aligned} \tag{2.4}$$

From (2.1), (2.3), and (2.4) the theorem follows.

3. That (2.1) is more general than (1.3) may be shown as follows. Suppose that (1.3) is satisfied. Then

$$\begin{aligned} \Psi(\alpha) &= \int_\alpha^\delta \frac{\Phi'(t)}{t}\, dt = \frac{\Phi(\delta)}{\delta} - \frac{\Phi(\alpha)}{\alpha} + \int_\alpha^\delta \frac{\Phi(t)}{t^2}\, dt \\ &= O(1) + o(1) + \int_\alpha^\delta o\left(\frac{1}{t}\right) dt \\ &= o\left(\log \frac{1}{\alpha}\right). \end{aligned}$$

Thus (2.1) is satisfied whenever (1.3) is satisfied.

On the other hand it is not difficult to choose $\phi(x)$ so as to satisfy (2.1) and *not* (1.3). Suppose for example that

$$\phi(\alpha) = 1 \quad \left(\frac{1}{p!} \leq \alpha \leq \frac{2}{p!},\ p = 1, 2, 3, \ldots\right),$$

and that $\phi(\alpha)$ is otherwise zero. If

$$\frac{2}{(q+1)!} < \alpha < \frac{1}{q!},$$

we have

$$\Psi(\alpha) = \sum_1^q \int_{1/p!}^{2/p!} \frac{dt}{t} = q \log 2,$$

$$\log \frac{1}{\alpha} > \log q! \sim q \log q,$$

$$q = O\left\{\left(\log \frac{1}{\alpha}\right) \Big/ \left(\log \log \frac{1}{\alpha}\right)\right\} = o\left(\log \frac{1}{\alpha}\right),$$

$$\Psi(\alpha) = o\left(\log \frac{1}{\alpha}\right);$$

while if

$$\frac{1}{(q+1)!} < x < \frac{2}{(q+1)!}$$

we have

$$\Psi(x) < (q+1) \log 2 = o\left(\log \frac{1}{\alpha}\right).$$

Thus (2.1) is satisfied in either case. But (1.3) is not satisfied, for, taking

$$\alpha = \frac{1}{(q+1)!},$$

we have

$$\Phi(\alpha) = \sum_{q+1}^{\infty} \frac{1}{p!} = \frac{1}{(q+1)!}\left(1 + \frac{1}{q+2} + \ldots\right)$$

$$\sim \frac{1}{(q+1)!} = \alpha.$$

4. The condition (2.1) is thus a genuine generalisation of (1.3). The interest of Theorem A lies, however, less in its superior generality than in its relations to other theorems concerning Fourier series, and in particular to Dini's classical theorem concerning their convergence. These relations are put in evidence by the following intermediate theorem.

THEOREM B. *Suppose that $\chi(x)$ is a function of x which tends steadily to infinity with x, and that*

$$\chi(x) = o(\log x). \tag{4.1}$$

$$\Psi(\alpha) = o\left\{\frac{\log(1/\alpha)}{\chi(1/\alpha)}\right\}. \tag{4.2}$$

Then

$$s_n = o\left\{\frac{\log n}{\chi(n)}\right\}. \tag{4.3}$$

There is a corresponding result in which o is replaced by O in (4.2) *and* (4.3).

As in § 2, we have

$$j_n = O\left\{n\Phi\left(\frac{1}{n}\right)\right\} + O\left\{\Psi\left(\frac{1}{n}\right)\right\}, \tag{4.4}$$

and

$$\Phi(\alpha) < \int_0^\alpha \Psi(t)\,dt \tag{4.5}$$

$$= o\int_0^\alpha \frac{\log(1/t)}{\chi(1/t)}\,dt = o\left\{\frac{1}{\chi(1/\alpha)}\int_0^\alpha \log\left(\frac{1}{t}\right)dt\right\}$$

$$= o\left\{\frac{\alpha\log(1/\alpha)}{\chi(1/\alpha)}\right\}.$$

From (4.2), (4.4), and (4.5) the theorem follows.

Theorem A and Dini's theorem correspond to the two extreme cases, just excluded by the conditions of Theorem B, the cases in which $\chi(x)=1$ and $\chi(x)=\log x$. In the latter case (4.2) and (4.3) must be replaced by

$$\Psi(\alpha) = H + o(1) \tag{4.6}$$

and

$$s_n = s + o(1) \tag{4.7}$$

respectively. The superiority of Theorem A over my original result lies in the fact that it is, so to say, a member of a continuous chain of theorems connecting it with Dini's theorem. We may say, roughly, that *the order of* s_n *is at most equal to that of*

$$\Psi\left(\frac{1}{n}\right) = \int_{1/n}^{\delta} \frac{|\phi(t)|}{t}\,dt.$$

5. Suppose that $\chi(x)$ satisfies the further condition

$$\Delta\frac{n\chi(n)}{\log n} = O\frac{\chi(n)}{\log n}.$$

This condition is certainly satisfied, for example, if

$$\chi(x) = (\log x)^\alpha (\log\log x)^\beta \ldots \quad (0<\alpha<1).$$

Then it is easily proved, by the argument which I used in my paper already referred to, that *the series*

$$\Sigma\frac{\chi(n)}{\log n}A_n$$

is convergent.

SIMPLE TYPES OF KAPTEYN SERIES.

By *G. N. Watson.*

1. It has been shewn by Kapteyn* that an arbitrary analytic function $f(z)$ can be expanded into a series of Bessel functions of the form $\sum_{m=0}^{\infty} a_m J_m(mz)$, provided that $f(z)$ is analytic when z lies in the region of the Argand diagram, for which the inequality

$$|z \exp \sqrt{(1-z^2)}| < |1 + \sqrt{(1-z^2)}|$$

is satisfied; and the expansion is valid throughout this region.

The region in question is the interior of a closed curve C, of which the longest diameter is the line joining the points ± 1 and the shortest is the line joining the points $\pm i \times 0{\cdot}663$.

The polar coordinates of any point of C are expressible in terms of a parameter by the equations

$$r = \sqrt{(2t \operatorname{cosech} 2t)}, \quad \cos 2\theta = \cosh 2t - t \sinh 2t.$$

In particular, the expansion of $(\frac{1}{2}z)^\nu$, when ν is a positive integer, is given by the formula

$$(\tfrac{1}{2}z)^\nu = \nu^2 \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}} J_{\nu+2s}\{(\nu+2s)z\}.$$

These results are obtained, as is to be expected, by means of the theory of contour integration. An elementary proof of the expansion of $(\frac{1}{2}z)^\nu$ has been given by Nielsen.† This proof, which is, in reality, a verification of Kapteyn's result, consists in expanding the Bessel functions in powers of z and verifying by a somewhat intricate and laborious process of induction that the coefficient of $z^{\nu+2s}$ vanishes except when $s=0$. This process of rearrangement is only necessarily legitimate when the series of moduli

$$\nu^2 \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}} I_{\nu+2s}\{(\nu+2s)|z|\}$$

is convergent, *i.e.* when‡ $|z| < 0{\cdot}663$.

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* *Ann. de l'École Norm. Sup.* (3), x. (1893), pp. 91–120.

† *Handbuch der Theorie der Cylinderfunktionen*, pp. 300–303.

‡ Nielsen proves by elementary methods that

$$\lim_{n \to \infty} \sqrt[n]{I_n(x)} < 1$$

where $0 \leq x < 0{\cdot}659$; but it is evident from Kapteyn's investigation (if ν is an integer) or from Debye's asymptotic expansion (*München. Sitzungsberichte*, v., 1910) that this number may be replaced by 0·663.

Nielsen's investigation is valid when ν is positive, though not necessarily an integer. So far as applications (*e.g.* such as those which occur in the theory of Electromagnetic Radiation) are concerned, the real interest of the series lies in the fact that it converges when z is real and $-1 \leq z \leq 1$, and *not* in the fact that it converges when z is a complex number such that $|z| < 0{\cdot}663$. From this point of view, Nielsen's investigation seems to be somewhat crude and artificial. The object of this note is to point out what seems to be the most natural way of investigating the expansion when ν is an integer. When ν is not an integer, the expansion is of a deeper type and more elaborate methods are required in investigating its validity. Some of the subsidiary results required in this paper are not new, but, in order to make the paper complete in itself, it seemed more desirable to give proofs of them than to give references to a number of scattered writings.

2. We shall first shew that,* if $\delta > 0$ and $0 \leq x \leq 1 - \delta$, then

$$J_n(nx) = O(n^{-p}),$$

when n is large, where p is any fixed positive number, whether n is an integer or not.

We take the Bessel-Schläfli† integral (valid whether n is an integer or not, provided that $x \geq 0$)

$$J_n(nx) = \frac{1}{\pi}\int_0^\pi \cos\{n(\theta - x\sin\theta)\}\,d\theta - \frac{\sin n\pi}{\pi}\int_0^\infty e^{-n\theta - nx\sinh\theta}\,d\theta,$$

and, on writing $\theta - x\sin\theta \equiv \phi$, $\theta + x\sinh\theta \equiv \chi$ and integrating by parts $2r$ times we get

$$J_n(nx) = \frac{1}{\pi}\left[\sum_{m=0}^{2r-1} \frac{d^{m+1}\theta}{d\phi^{m+1}} \cdot \frac{\sin(n\phi + \frac{1}{2}m\pi)}{n^{m+1}}\right]_0^\pi + \frac{(-)^r}{\pi n^{2r}}\int_0^\pi \cos n\phi\,\frac{d^{2r+1}\theta}{d\phi^{2r+1}}\,d\phi$$

$$+ \frac{\sin n\pi}{\pi}\left[\sum_{m=0}^{2r-1} \frac{d^{m+1}\theta}{d\chi^{m+1}} \cdot \frac{(-)^m e^{-nx}}{n^{m+1}}\right]_0^\infty - \frac{\sin n\pi}{\pi n^{2r}}\int_0^\infty e^{-nx}\frac{d^{2r+1}\theta}{d\chi^{2r+1}}\,d\chi.$$

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* Special cases of this result have long been known, see *e.g.* Routh, *Dynamics of a Particle*, p. 299.

† The Bessel-Schläfli integral is usually obtained by using contour integrals (see Whittaker and Watson, *Modern Analysis*, p. 356), but when n is an integer the integral reduces to Bessel's integral, which is obtainable by elementary methods.

Since $\frac{d}{d\phi} = (1 - x\cos\theta)^{-1}\frac{d}{d\theta}$, it is easy to see that $\frac{d^{m+1}\theta}{d\phi^{m+1}}$ is a rational function of $\sin\theta$ and $\cos\theta$, which is bounded since $(1 - x\cos\theta)^{-1} \leq 1/\delta$, and therefore, having a finite number of maxima and minima in the range $(0, \pi)$, it has limited total fluctuation. Also the first set of integrated terms vanish at the lower limit, and alternate members of the set contain $\sin\theta$ as a factor and therefore also vanish at the upper limit.

In like manner alternate members of the second set of integrated terms vanish at the lower limit, since they contain $\sinh\theta$ as a factor, and the remainder cancel with the alternate members of the first set at the upper limit. This is easily seen, either by considering the construction of the Bessel-Schläfli integral from a contour integral, or by observing that $\pi + i\chi$ is the same function of $\pi + i\theta$ as ϕ is of θ.

The second set of integrated terms vanish at the upper limit, and $(d^{2r+1}\theta/d\chi^{2r+1})$ is easily seen to be a bounded function of θ.

Let the upper bounds of $\frac{d^{2r+1}\theta}{d\phi^{2r+1}}$, $\frac{d^{2r+1}\theta}{d\chi^{2r+1}}$, be A_r, B_r respectively, and we see at once that

$$|J_n(nx)| \leq \frac{A_r}{\pi n^{2r}}\int_0^\pi |\cos n\phi|\, d\phi + \frac{B_r}{\pi n^{2r}}\int_0^\infty e^{-n\chi}\, d\chi$$

$$\leq A_r n^{-2r} + B_r \pi^{-1} n^{-2r-1}.$$

Since A_r, B_r are independent of n, and r is *any* fixed integer, we see at once that, if p is *any* positive number,

$$J_n(nx) = O(n^{-p}).$$

Since $J_n(-nx) = e^{\pm n\pi i} J_n(nx)$, we see that this result is true throughout the range $-1 + \delta \leq x \leq 1 - \delta$.

In like manner, since

$$J_n'(nx) = \frac{1}{\pi}\int_0^\pi \sin\theta \sin n\phi \frac{d\theta}{d\phi}\, d\phi + \frac{\sin n\pi}{\pi}\int_0^\infty \sinh\theta\, e^{-n\chi}\frac{d\theta}{d\chi}\, d\chi$$

when $0 \leq x \leq 1 - \delta$, and since

$$\frac{d^m}{d\phi^m}\left\{\sin\theta\frac{d\theta}{d\phi}\right\}, \quad e^{-\theta}\frac{d^m}{d\chi^m}\left\{\sinh\theta\frac{d\theta}{d\chi}\right\}$$

are bounded functions of θ, we see that, when $-1 + \delta \leq x \leq 1 - \delta$,

$$J_n'(nx) = O(n^{-p}),$$

and similarly $$J_n''(nx) = O(n^{-p}).$$

It follows at once that, if $a_s = O(s^k)$, then the series

$$\sum_{s=0}^{\infty} a_s J_{\nu+s}\{(\nu+s)x\}$$

is uniformly convergent throughout the range

$$-1+\delta \leq x \leq 1-\delta,$$

and it may be differentiated term-by-term any number (m) of times; for it is sufficient to take $p > m+s+2$ to see that the general term of the differentiated series is $O(s^{-2})$, where the constant implied in the symbol O depends on δ but does *not* depend on x; so the differentiated series converges uniformly, by Weierstrass' test, and therefore the term-by-term differentiation is permissible.

3. We are now in a position to establish the induction from which we derive the expansion of x^ν into a Kapteyn series.

The induction depends on the fact that, if we know the sum, $f(x)$, of the Kapteyn series $\sum_{m=1}^{\infty} a_m J_m(mx)$, we can find the sum, $F(x)$, of the Kapteyn series $\sum_{m=1}^{\infty} a_m m^{-2} J_m(mx)$, by solving a linear differential equation of the second order, provided that a_m is $O(m^k)$ for some fixed index k.

For, since $J_m(mx)$ satisfies the differential equation*

$$m^2x^2 J_m''(mx) + mx J_m'(mx) + (m^2x^2 - m^2) J_m(mx) = 0,$$

we have

$$\begin{aligned} x^2 F''(x) + xF'(x) &= \sum_{m=1}^{\infty} a_m \left[x^2 J_m''(mx) + xm^{-1} J_m'(mx)\right] \\ &= (1-x^2) f(x), \end{aligned}$$

so that $F(x)$ satisfies the differential equation

$$x^2 F''(x) + xF'(x) = (1-x^2) f(x).$$

Now suppose that, for some particular value of ν, x^ν is expansible in the form

$$x^\nu = \nu^2 \sum_{m=1}^{\infty} b_{m,\nu} J_m(mx),$$

where $b_{m,\nu}$ is $O(m^k)$.

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* As usual, we use primes to denote differentiations with regard to the argument of the function; thus

$$J_m'(mx) = \frac{1}{m}\frac{d}{dx} J_m(mx), \quad F'(x) = \frac{d}{dx} F(x).$$

Consider the function

$$\phi(x) \equiv (\nu+2)^2 \sum_{m=1}^{\infty} \frac{m^2-\nu^2}{m^2} b_{m,\nu} J_m(mx).$$

Then

$$x^2\phi''(x)+x\phi'(x) = (1-x^2)(\nu+2)^2 \sum_{m=1}^{\infty} (m^2-\nu^2) b_{m,\nu} J_m(mx)$$

$$= (\nu+2)^2 \left\{x^2 \frac{d^2}{dx^2} + x \frac{d}{dx}\right\} \frac{x^\nu}{\nu^2} - (1-x^2)(\nu+2)^2 x^\nu$$

$$= (\nu+2)^2 x^{\nu+2}.$$

Solving the differential equation

$$x^2\phi''(x)+x\phi'(x) = (\nu+2)^2 x^{\nu+2},$$

we see that

$$\phi(x) = x^{\nu+2} + A + B\log x,$$

where A and B are constants. Making $x \to 0$, and using the fact that the series for $\phi(x)$ is continuous throughout the range $-1+\delta \leq x \leq 1-\delta$, we see that $A=B=0$.

Hence, if $$x^\nu = \nu^2 \sum_{m=1}^{\infty} b_{m,\nu} J_m(mx),$$

then $$x^{\nu+2} = (\nu+2)^2 \sum_{m=1}^{\infty} b_{m,\nu+2} J_m(mx),$$

where $$b_{m,\nu+2} = \frac{m^2-\nu^2}{m^2} b_{m,\nu}.$$

Hence, if we can obtain the Kapteyn series for x and x^2, we obtain by this induction the Kapteyn series for x^ν, where ν is any positive integer.

4. The Kapteyn series for x and x^2 are already known,* but, for the sake of completeness, we shall now obtain them anew.

If $\phi = \theta - x\sin\theta$, one branch of θ is a continuous monotonic one-valued odd function of ϕ, vanishing with ϕ, provided that $-1+\delta \leq x \leq 1-\delta$; and so $(1-x\cos\theta)^{-1}$ may be expanded in a Fourier cosine-series, thus

$$(1-x\cos\theta)^{-1} = \tfrac{1}{2}c_0 + \sum_{m=1}^{\infty} c_m \cos m\phi,$$

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* Cf. Routh, *Dynamics of a Particle*, p. 300.

where $$\pi c_m = \int_{-\pi}^{\pi} \cos m\phi \, (1 - x\cos\theta)^{-1} d\phi,$$

$$= 2\pi J_m(mx), \qquad 2\pi,$$

according as $m \neq 0$ or $m = 0$.

Putting in turn $\phi = 0$, $\phi = \pi$, we get

$$(1-x)^{-1} = 1 + 2\sum_{m=1}^{\infty} J_m(mx),$$

$$(1+x)^{-1} = 1 + 2\sum_{m=1}^{\infty} (-)^m J_m(mx).$$

Hence, using the differential equation of § 3, we see that, if

$$\sum_{m=1}^{\infty} m^{-2} J_m(mx) = g_1(x), \quad \sum_{m=1}^{\infty} (-)^m m^{-2} J_m(mx) = g_2(x),$$

then $$x^2 g_1''(x) + x g_1'(x) = (1-x^2)\sum_{m=1}^{\infty} J_m(mx)$$

$$= \tfrac{1}{2}x(1+x),$$

and so $$g_1(x) = \tfrac{1}{4}x + \tfrac{1}{8}x^2,$$

and similarly $$g_2(x) = -\tfrac{1}{4}x + \tfrac{1}{8}x^2,$$

the constants of integration vanishing, as in the corresponding analysis of § 3.

Comparing these results we see that

$$x^2 = 2\sum_{m=1}^{\infty} \frac{1}{m^2} J_{2m}(2mx), \quad x = 2\sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} J_{2m+1}\{(2m+1)x\},$$

so that $$b_{2m+1,2} = 0, \quad b_{2m,2} = 1/(2m^2).$$

Hence, from the recurrence-formulæ, we see that, *when ν is even*, $b_{2m+1,\nu} = 0$ and

$$b_{2m,\nu} = \frac{1}{2m^2} \frac{\{(2m)^2 - 2^2\}\{(2m)^2 - 4^2\}\ldots\{(2m)^2 - (\nu-2)^2\}}{(2m)^{\nu-2}}$$

$$= \frac{\Gamma(m + \frac{1}{2}\nu)}{2m^{\nu+1}\Gamma(m - \frac{1}{2}\nu + 1)}.$$

Consequently we have

$$(\tfrac{1}{2}x)^\nu = \nu^2 \sum_{m=1}^{\infty} \frac{\Gamma(m + \frac{1}{2}\nu)}{2^{\nu+1} m^{\nu+1} \Gamma(m - \frac{1}{2}\nu + 1)} J_{2m}(2mx)$$

$$= \nu^2 \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}} J_{\nu+2s}\{(\nu+2s)x\},$$

on writing $m-\frac{1}{2}\nu=s$ and noticing that the first $\frac{1}{2}\nu$ terms of the original series vanish.

We obtain the same result, *when ν is odd*, from the expansion

$$\tfrac{1}{2}x=\sum_{s=0}^{\infty}\frac{\Gamma(1+s)}{s!\,(1+2s)^2}J_{1+2s}\{(1+2s)x\},$$

which has been given above; as the analysis does not differ appreciably from that appropriate to the case when ν is even, it seems unnecessary to give it.

5. In order to establish the expansion for all *positive* values of ν (not necessarily integers), it seems simplest to use a slight modification of Nielsen's method.

The coefficient of $x^{\nu+2r}$ in the expansion

$$\sum_{s=0}^{\infty}\frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}}J_{\nu+2s}\{(\nu+2s)x\}$$

is

$$\sum_{s=0}^{r}\frac{\Gamma(\nu+s)}{(\nu+2s)^{\nu+1}s!}\,\frac{(-)^{r-s}(\nu+2s)^{\nu+2r}}{2^{\nu+2r}\Gamma(r-s+1)\,\Gamma(\nu+r+s+1)}$$

$$=\frac{\Gamma(\nu)}{2^{\nu+2r}\Gamma(\nu+2r+1)}\sum_{s=0}^{r}\frac{(-)^{r-s}(\nu+2s)^{2r-1}}{s!\,(r-s)!}\,\frac{\Gamma(\nu+s)}{\Gamma(\nu)}\,\frac{\Gamma(\nu+2r+1)}{\Gamma(\nu+r+s+1)}.$$

The last sum (when $r>0$) is a polynomial of degree $3r-1$ in ν which vanishes whenever ν is an integer; consequently it is *identically zero* for all values of ν when $r>0$.

The implied rearrangement of series is justifiable, as proved by Nielsen, when $|x|<0{\cdot}663$. For other values of x in the interior of C the result may be deduced either by Kapteyn's process of contour integration, or by the theory of analytic continuation.

It is, of course, only to be expected that the expansion should need a proof of a deeper nature when ν is not restricted to be a positive integer.

6. To obtain the expansion when $x=\pm 1$ we observe that, whenever $x\geq 0$,

$$|J_n(x)|=\left|\frac{1}{\pi}\int_0^{\pi}\cos n\phi\,d\theta-\frac{\sin n\pi}{\pi}\int_0^{\infty}e^{-n\chi}\frac{d\theta}{d\chi}d\chi\right|$$

$$\leq 1+\pi^{-1}\int_0^{\infty}e^{-n\chi}d\chi,$$

since $d\theta/d\chi \leq 1$; and so, whether x is positive or negative,

$$|J_n(x)| \leq 1 + 1/(n\pi).$$

Hence, by Weierstrass' test, the expansion of $(\frac{1}{2}x)^\nu$ is uniformly convergent* (and therefore a continuous function of x) throughout the range $-1 \leq x \leq 1$; and so

$$\lim_{x \to \pm 1} (\tfrac{1}{2}x)^\nu = \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}} \lim_{x \to \pm 1} J_{\nu+2s}\{(\nu+2s)x\},$$

which gives

$$\frac{1}{2^\nu} = \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)}{s!\,(\nu+2s)^{\nu+1}} J_{\nu+2s}(\nu+2s).$$

The sum of the series is convergent and continuous, even when $x \geq 1$, but its sum is not $(\frac{1}{2}x)^\nu$ when $x > 1$. Of course there are an unlimited number of functions which are *continuous* for all real values of x and equal to $(\frac{1}{2}x)^\nu$ when $-1 \leq x \leq 1$, and not equal to it when $x > 1$.

I propose to deal with the sums of various Kapteyn series, in which $x > 1$, in a subsequent paper.

A RAPIDLY CONVERGENT SERIES FOR $\log N!$

By *Prof. W. Burnside.*

ALTHOUGH the divergent series for $\log N!$ given by Stirling's theorem cannot be bettered for purposes of numerical calculation, a rapidly convergent series, which can be readily established, possesses some theoretical interest. Consider the area bounded by the axes and the hyperbola

$$2xy + x + y = N.$$

It is given by

$$\int_0^N \frac{N-x}{2x+1}\,dx = \tfrac{1}{2}(N+\tfrac{1}{2})\log(2N+1) - \tfrac{1}{2}N.$$

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* The general term of the expansion is $O(s^{-2})$.

The area between the two hyperbolas

$$2xy + x + y = n - 1, \quad 2xy + x + y = n,$$

and the axes is therefore

$$\tfrac{1}{2}\left[(n+\tfrac{1}{2})\log(2n+1) - (n-\tfrac{1}{2})\log(2n-1) - 1\right];$$

and by dividing the original area into strips in this way it ollows that

$$\sum_1^N \left[n \log \frac{2n+1}{2n-1} + \tfrac{1}{2}\log(4n^2-1) - 1 \right] = (N+\tfrac{1}{2})\log(2N+1) - N.$$

Now

$$n \log \frac{2n+1}{2n-1} + \tfrac{1}{2}\log(4n^2-1) - 1$$

$$= \log n + \log 2 - \sum_1^\infty \frac{1}{2r(2r+1)2^{2r}n^{2r}}.$$

Hence

$$\sum_1^N \left[\log n + \log 2 - \sum_1^\infty \frac{1}{2r(2r+1)2^{2r}n^{2r}} \right] = (N+\tfrac{1}{2})\log(2N+1) - N,$$

or

$$\log N! = (N+\tfrac{1}{2})\log(N+\tfrac{1}{2}) - N + \tfrac{1}{2}\log 2 + \sum_{r=1}^\infty \sum_{n=1}^N \frac{1}{2r(2r+1)2^{2r}n^{2r}}.$$

If the constant

$$\tfrac{1}{2}\log 2 + \sum_1^\infty \sum_1^\infty \frac{1}{2r(2r+1)2^{2r}n^{2r}}$$

be denoted by C, the formula is

$$\log N! = (N+\tfrac{1}{2})\log(N+\tfrac{1}{2}) - N + C - \sum_{r=1}^\infty \sum_{n=N+1}^\infty \frac{1}{2r(2r+1)2^{2r}n^{2r}}.$$

The series on the right is rapidly convergent for even moderate values of N. In fact $\sum_{N+1}^\infty \frac{1}{n^{2r}}$ is less than, but if N is large, very nearly equal to, $\frac{1}{(2r-1)} \frac{1}{(N+\frac{1}{2})^{2r-1}}$; so that the series converges more rapidly than

$$\sum_{r=1}^\infty \frac{1}{(2r-1)2r(2r+1)2^{2r}(N+\frac{1}{2})^{2r-1}}.$$

The calculated value of C, to six figures, as defined above, is .418938; but its actual value is determined at once by comparing the above formula with that given by Stirling's theorem. Thus it follows from the above formula that

$$\operatorname*{Lt}_{N=\infty} \frac{e^C (N+\frac{1}{2})^{N+\frac{1}{2}} e^{-N}}{N!} = 1.$$

Now the formula obtained from Stirling's theorem is

$$\operatorname*{Lt}_{N=\infty} \frac{\sqrt{(2\pi N)}.N^N e^{-N}}{N!} = 1.$$

Hence

$$\operatorname*{Lt}_{N=\infty} \frac{e^C (N+\frac{1}{2})^{N+\frac{1}{2}}}{\sqrt{(2\pi N)}.N^N} = 1,$$

which gives

$$e^C = \sqrt{\left(\frac{2\pi}{e}\right)}.$$

The formula for $\log N!$ may therefore be written

$$\log N! = (N+\tfrac{1}{2}) \log (N+\tfrac{1}{2}) - N + \log \sqrt{\left(\frac{2\pi}{e}\right)} - \sum_{r=1}^{r=\infty} \sum_{n=N+1}^{\infty} \frac{1}{2r(2r+1)2^{2r} n^{2r}}.$$

AN APPROXIMATE EXPRESSION FOR $\sum_{n=0}^{\infty} \frac{1}{(m+n)^r}$.

By *Prof. W. Burnside.*

For all values of m and x, such that zero does not lie between m and $m+x$, the relation

$$\frac{1}{(m+x)^r} = \frac{1}{m^r} - r\frac{x}{m^{r+1}} + \frac{r.r+1}{1.2} \cdot \frac{x^2}{m^{r+2}} - \frac{r.r+1.r+2}{3!} \frac{x^3}{m^{r+3}}$$

$$+ \frac{r.r+1.r+2.r+3}{4!} \frac{x^4}{m^{r+4}} - \frac{r.r+1.r+2.r+3.r+4}{5!} \frac{x^5}{m^{r+5}}$$

$$+ \frac{r(r+1)(r+2)(r+3)(r+4)(r+5)}{5!} \int_0^x \frac{(t-x)^5}{(m+t)^{r+6}} dt$$

is an identity.

Integrating it for values of x from $-\frac{1}{2}$ to $\frac{1}{2}$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{(m+x)^r} = \frac{1}{m^r} + \frac{r(r+1)}{24}\frac{1}{m^{r+2}} + \frac{r(r+1)(r+2)(r+3)}{2^4.5!}\frac{1}{m^{r+4}}$$

$$+ \frac{r(r+1)(r+2)(r+3)(r+4)(r+5)}{5!}\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_0^x dt \frac{(t-x)^5}{(m+t)^{r+6}}.$$

Similarly

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{(m+x)^{r+2}} = \frac{1}{m^{r+2}} + \frac{(r+2)(r+3)}{24}\frac{1}{m^{r+4}}$$

$$+ \frac{(r+2)(r+3)(r+4)(r+5)}{3!}\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_0^x dt \frac{(t-x)^3}{(m+t)^{r+6}},$$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{(m+x)^{r+4}} = \frac{1}{m^{r+4}} + (r+4)(r+5)\int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_0^x dt \frac{t-x}{(m+t)^{r+6}}.$$

Eliminating $\frac{1}{m^{r+2}}$ and $\frac{1}{m^{r+4}}$ between these equations, there results

$$\frac{1}{m^r} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left\{ \frac{1}{(m+x)^r} - \frac{r(r+1)}{24}\frac{1}{(m+x)^{r+2}} \right.$$

$$\left. + \frac{7}{8}\frac{r(r+1)(r+2)(r+3)}{6!}\frac{1}{(m+x)^{r+4}} \right\} dx$$

$$- \frac{r(r+1)(r+2)(r+3)(r+4)(r+5)}{6!} R,$$

where

$$R = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_0^x dt \frac{6(t-x)^5 - 5(t-x)^3 + \frac{7}{8}(t-x)}{(m+t)^{r+6}}.$$

Writing $f(x, t)$ for the integrand in R,

$$R = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_0^x dt f(x, t)$$

$$= \int_0^{\frac{1}{2}} dx \int_0^x dt [f(x, t) - f(-x, -t)]$$

$$= \int_0^{\frac{1}{2}} dt \int_t^{\frac{1}{2}} dx [f(x, t) - f(-x, -t)]$$

$$= \int_0^{\frac{1}{2}} dt \left\{ -(t-\tfrac{1}{2})^6 + \tfrac{5}{4}(t-\tfrac{1}{2})^4 - \tfrac{7}{16}(t-\tfrac{1}{2}) \right\} \left\{ \frac{1}{(m+t)^{r+6}} + \frac{1}{(m-t)^{r+6}} \right\}.$$

The function in the first bracket lies between 0 and $-\frac{3}{64}$ when t is between 0 and $\frac{1}{2}$. Hence if θ' is a positive proper fraction

$$R = -\theta' \frac{3}{64} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dt}{(m+t)^{r+6}}.$$

Hence, if m is greater than $\frac{1}{2}$,

$$\frac{1}{m^r} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{1}{(m+x)^r} - \frac{r(r+1)}{24} \frac{1}{(m+x)^{r+2}} + \frac{7}{8} \frac{r(r+1)(r+2)(r+3)}{6!} \frac{1}{(m+x)^{r+4}} \right] dx.$$
$$+ \theta' \frac{3}{2^6} \frac{r(r+1)(r+2)(r+3)(r+4)(r+5)}{6!} \frac{1}{(m+x)^{r+6}} \Bigg]$$

Now if $r>1$, the series $\sum_0^\infty \frac{1}{(m+n)^r}$ and the integral $\int_{-\frac{1}{2}}^{\infty} \frac{dx}{(m+x)^r}$ are both convergent. Hence

$$\sum_{m=0}^{\infty} \frac{1}{(m+n)^r} = \frac{1}{(r-1)(m-\frac{1}{2})^{r-1}} - \frac{r}{24} \frac{1}{(m-\frac{1}{2})^{r+1}}$$
$$+ \frac{7}{8} \frac{r(r+1)(r+2)}{6!} \frac{1}{(m-\frac{1}{2})^{r+3}}$$
$$+ \theta \frac{3}{2^6} \frac{r(r+1)(r+2)(r+3)(r+4)}{6!} \frac{1}{(m-\frac{1}{2})^{r+5}},$$

where θ is a positive proper fraction.

Unless r is comparatively large, in which case the series is rapidly convergent, the remainder term in this formula will be very small for quite moderate values of m. Thus for $r=2$, $m=7$, it is not more than one unit in the eighth decimal place. For the case in which $r-1$ is small, so that the series is very slowly convergent, the proportional error involved in neglecting the remainder term is excessively small for quite moderate values of m. For instance, when $r=1.01$, $m=3$, the proportional error due to neglecting the remainder term is less than 3×10^{-7}.

AN EXPRESSION FOR THE REMAINDER IN THE EXPANSION OF A FUNCTION IN A SERIES OF POSITIVE INTEGRAL POWERS OF THE VARIABLES.

By *Prof. W. Burnside.*

1. Let $f(x)$ be a function of the real variable x, which, with its differential coefficients up to the $(m+1)^{\text{th}}$, is finite, continuous, and one-valued, when $-a \leq x \leq a$. Then, if x lies between these limits,

$$\int_0^x \frac{(x-u)^m}{m!} f^{m+1}(u)\, du = -\frac{x^m}{m!} f^m(0) + \int_0^x \frac{(x-u)^{m-1}}{m-1!} f^m(u)\, du,$$

and continuing the integration by parts,

$$\int_0^x \frac{(x-u)^m}{m!} f^{m+1}(u)\, du = -\frac{x^m}{m!} f^m(0) - \frac{x^{m-1}}{m-1!} f^{m-1}(0) - \ldots - xf'(0) + f(x) - f(0),$$

or

$$f(x) = f(0) + xf'(0) + \ldots + \frac{x^m}{m!} f^m(0) + \int_0^x \frac{(x-u)^m}{m!} f^{m+1}(u)\, du \ldots (1).$$

Under the conditions assumed with respect to $f(x)$ this relation is, in fact, an identity.

2. Let $f(x, y)$ be a function of the two real variables x and y, such that it and its partial differential coefficients $\dfrac{\partial^{r+s} f}{\partial x^r y^s}$ $(r \leq m+1,\ s \leq n+1)$ are one-valued, finite, and continuous when $-a \leq x \leq a,\ -b \leq y \leq b$. If x and y lie between these limits,

$$I_{m,n} = \int_0^x du \int_0^y dv\, \frac{(x-u)^m (y-v)^n}{m!\, n!} \frac{\partial^{m+n+2} f(u, v)}{\partial u^{m+1} \partial v^{n+1}}$$

$$= \int_0^x du \left[\frac{(x-u)^m (y-v)^n}{m!\, n!} \frac{\partial^{m+n+1} f}{\partial u^{m+1} dv^n} \right]_0^y + \int_0^x du \int_0^y dv\, \frac{(x-u)^m (y-v)^{n-1}}{m!\, n-1!} \frac{\partial^{m+n+1} f}{\partial u^{m+1} \partial v^n},$$

or

$$I_{m,n} - I_{m,n-1} = -\frac{y^n}{n!} \int_0^x \frac{(x-u)^m}{m!} \left(\frac{\partial^{m+n+1} f}{\partial u^{m+1} \partial v^n} \right)_{v=0} du.$$

If
$$\left(\frac{\partial^n f(u, v)}{\partial v^n}\right)_{v=0} = F_n(v),$$

then, by equation (1),

$$I_{m,n} - I_{m,n-1} = -\frac{y^n}{n!}\left[F_n(x) - F_n(0) - xF_n'(0) - \ldots - \frac{x^m}{m!}F_n^{m}(0)\right].$$

Now
$$F_n^{r}(0) = \frac{\partial^{r+n} f(x, y)}{\partial x^r \partial y^n}_{x=0, y=0} = \frac{\partial^{r+n} f(0, 0)}{\partial x^r \partial y^n} \text{ say,}$$

and
$$F_n(x) = \frac{\partial^n f(x, y)}{\partial y^n}_{y=0} = \frac{\partial^n f(x, 0)}{\partial y^n}.$$

Hence
$$I_{m,n} - I_{m,n-1} = -\frac{y^n}{n!}\left[\frac{\partial^n f(x, 0)}{\partial y^n} - \frac{\partial^n f(0, 0)}{\partial y^n} - \sum_1^m \frac{x^r}{r!}\frac{\partial^{r+n} f(0, 0)}{\partial x^r \partial y^n}\right].$$

Similarly
$$I_{m,n-1} - I_{m,n-2} = -\frac{y^{n-1}}{n-1!}\left[\frac{\partial^{n-1} f(x, 0)}{\partial y^{n-1}} - \frac{\partial^{n-1} f(0, 0)}{\partial y^{n-1}} - \sum_1^m \frac{x^r}{r!}\frac{\partial^{r+n-1} f(0, 0)}{\partial x^r \partial y^{n-1}}\right],$$

and so on. Also

$$I_{m,1} = \int_0^x du \left[\frac{(x-u)^m (y-v)}{m!}\frac{\partial^{m+2} f(u, v)}{\partial u^{m+1} \partial v}\right]_{v=0}^{v=y} + \int_0^x du \int_0^y dv \frac{(x-u)^m}{m!}\frac{\partial^{m+2} f(u, v)}{\partial u^{m+1} \partial v}$$

$$= -y\left[\frac{\partial f(x, 0)}{\partial y} - \frac{\partial f(0, 0)}{\partial y} - \sum_1^m \frac{x^r}{r!}\frac{\partial^{r+1} f(0, 0)}{\partial x^r \partial y}\right] - \left[f(x, 0) - f(0, 0) - \sum_1^m \frac{x^r}{r!}\frac{\partial^r f(0, 0)}{\partial x^r}\right] - \int_0^x du \frac{(x-u)^m}{m!}\frac{\partial^{m+1} f(u, y)}{\partial u^{m+1}}.$$

Hence, adding the expressions for

$$I_{m,n} - I_{m,n-1},\ I_{m,n-1} - I_{m,n-2} \ldots, \text{ and } I_{m,1}$$

$$I_{m,n}=f(0,0)+\sum_{\substack{r\leq m\\ s\leq n}}\frac{x^r}{r!}\frac{y^s}{s!}\frac{\partial^{r+s}f(0,0)}{\partial x^r\partial y^s}+\int_0^x du\,\frac{(x-u)^m}{m!}\frac{\partial^{m+1}f(u,y)}{\partial u^{m+1}}-\left[f(x,0)+\sum_1^n\frac{y^s}{s!}\frac{\partial^s f(x,0)}{\partial y^s}\right].$$

Now from (i) the series in the last square bracket is

$$f(x,y)-\int_0^y dv\,\frac{(y-v)^n}{n!}\frac{\partial^{n+1}f(x,v)}{\partial v^{n+1}}.$$

Hence finally

$$f(x,y)=f(0,0)+\sum_{\substack{r\leq m\\ s\leq n}}\frac{x^r}{r!}\frac{y^s}{s!}\frac{\partial^{r+s}f(0,0)}{\partial x^r\partial y^s}+\int_0^x du\,\frac{(x-u)^m}{m!}\frac{\partial^{m+1}f(u,y)}{\partial u^{m+1}}+\int_0^y dv\,\frac{(y-v)^n}{n!}\frac{\partial^{n+1}f(x,v)}{\partial v^{n+1}}-\int_0^x du\int_0^y dv\,\frac{(x-u)^m(y-v)^n}{m!\,n!}\frac{\partial^{m+n+2}f(u,v)}{\partial u^{m+1}\partial v^{n+1}}.$$

3. In the same way it may be shewn that, under similar conditions,

$$f(x,y,z)=f(0,0,0)+\sum_{r\leq m,\,s\leq n,\,t\leq p}\frac{x^r}{r!}\frac{y^s}{s!}\frac{z^t}{t!}\frac{\partial^{r+s+t}f(0,0,0)}{\partial x^r\partial y^s\partial z^t}+\int_0^x du\,\frac{(x-u)^m}{m!}\frac{\partial^{m+1}f(u,y,z)}{\partial u^{m+1}}+\int_0^y dv\,\frac{(y-v)^n}{n!}\frac{\partial^{n+1}f(x,v,z)}{\partial v^{n+1}}+\int_0^z dw\,\frac{(z-w)^p}{p!}\frac{\partial^{p+1}f(x,y,w)}{\partial w^{p+1}}-\int_0^y dv\int_0^z dw\,\frac{(y-v)^n(z-w)^p}{n!\,p!}\frac{\partial^{n+p+2}f(x,v,w)}{\partial v^{n+1}\partial w^{p+1}}-\int_0^z dw\int_0^x du\,\frac{(z-w)^p(x-u)^m}{p!\,m!}\frac{\partial^{p+m+2}f(u,y,w)}{\partial u^{m+1}\partial w^{p+1}}-\int_0^x du\int_0^y dv\,\frac{(x-u)^m(y-v)^n}{m!\,n!}\frac{\partial^{m+n+2}f(u,v,z)}{\partial u^{m+1}\partial v^{n+1}}+\int_0^x du\int_0^y dv\int_0^z dw\,\frac{(x-u)^m(y-v)^n(z-w)^p}{m!\,n!\,p!}\frac{\partial^{m+n+p+3}f(u,v,w)}{\partial u^{m+1}\partial v^{n+1}\partial w^{p+1}},$$

and the formulæ are, in fact, quite general.

4. The limitation to real values of the variables is clearly not essential. If the function and all its partial differential coefficients up to the highest included in the formulæ are finite, continuous, and one-valued for all values of the variables, real or complex, subject to the conditions $|x|<a$, $|y|<b$, $|z|<c$, ..., and if the x, y, z, ... of the formulæ and the paths of integration lie within these ranges, the formulæ are still identities.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

XLVI.

On Stieltjes' 'problème des moments'.

1. The *problème des moments* is that of finding a function $f(x)$ such that

$$\int_0^\infty x^n f(x)\,dx = c_n \tag{1.1}$$

for $n=0, 1, 2, \ldots$, the c's being given constants. The problem was first attacked by Stieltjes*, who supposed $f(x)$ *positive*, and obtained the complete solution† in this case by means of some singularly beautiful analysis based on the theory of continued fractions.

Stieltjes proved in particular that there cannot be two different continuous functions $f_1(x)$ and $f_2(x)$, each of which satisfies the equations (1.1) and the inequalities

$$0 \leq f(x) < e^{-k\sqrt{x}}, \tag{1.2}$$

where k is a positive constant; and Borel showed that (1.2) may be replaced by

$$|f(x)| < e^{-k\sqrt{x}}. \tag{1.3}$$

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* T. J. Stieltjes, 'Recherches sur les fractions continues', *Annales de la Faculté des Sciences de Toulouse*, ser. 1, vol. viii. (1894), J., pp. 1–122, and vol. ix. (1895), A., pp. 1–47. See also E. Le Roy, 'Sur les séries divergentes et les fonctions définies par un développement de Taylor', *ibid.*, ser. 2, vol. ii. (1900), pp. 317–431: H. Lebesgue, 'Sur les intégrales singulières', *ibid.*, ser. 3, vol. i. (1909), pp. 25–111 (pp. 101 *et seq.*): E. Borel, *Leçons sur les séries divergentes* (1901), ch. 3: O. Perron, *Die Lehre von den Kettenbrüchen* (1913), ch. 9

† It is necessary, in order to obtain such a solution, to develop a theory of 'Stieltjes integrals' of the form $\int x^n d\psi(x)$, where $\psi(x)$ is a monotonic but not necessarily continuous function. I shall not be concerned in this note with such generalisations of the notion of an integral.

Thus no continuous function which satisfies (1.3) can also satisfy

$$\int_0^\infty x^n f(x)\,dx = 0 \tag{1.4}$$

for $n = 0, 1, 2, \ldots$.

This last theorem is certainly untrue if enunciated without some limitation on the order of $f(x)$. Thus Stieltjes gave the examples

$$f(x) = e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}, \quad f(x) = e^{-(\log x)^2} \sin(2\pi \log x),$$

both of which satisfy (1.4) for all positive integral values of n. I may observe in passing that, since

$$\int_0^\infty e^{-x^\mu \cos\alpha} \sin(x^\mu \sin\alpha)\,dx = \frac{1}{\mu}\,\Gamma\left(\frac{n+1}{\mu}\right) \sin\frac{(n+1)\alpha}{\mu}$$

if $\mu > 0$ and $0 < \alpha < \frac{1}{2}\pi$, the function

$$f(x) = e^{-x^\mu \cos\mu\pi} \sin(x^\mu \sin\mu\pi)$$

satisfies the equations (1.4) for every value of μ less than $\frac{1}{2}$. It is therefore impossible that (1.3) should be replaced by any condition

$$|f(x)| < e^{-kx^\mu},$$

where $\mu < \frac{1}{2}$*.

It has been suggested by Lebesgue† that it should be possible to prove some at any rate of Stieltjes' results, and in particular this theorem concerning the uniqueness of the solution, by methods independent of the theory of continued fractions. But the argument indicated by Lebesgue applies only to functions such that the integral

$$\int_0^\infty |f(x)|\, e^{Ax^2} dx$$

is convergent for every value of A.

In this note I suppose that the integral

$$\int_0^\infty |f(x)|\, e^{k\sqrt{x}} dx \tag{1.5}$$

is convergent for some positive k, and I give two different proofs of the uniqueness of the solution. Both are indepen-

* It is clear that Stieltjes (though he does not state the result explicitly) was aware that (1.2) could not be replaced by any inequalities of the form

$$0 \leq f(x) < e^{-kx^\mu} \quad (\mu < \tfrac{1}{2}).$$

It is probable that he knew the example that I have just given.

† *Loc. cit.*, p. 104.

dent of the theory of continued fractions. In each of the proofs I obtain incidentally a formula for the solution which is valid whenever a solution satisfying the condition exists. One of these formulæ is that given by Le Roy*, who established its validity in certain special cases. I also consider some other forms of the solution valid under narrower restrictions.

I impose no condition on $f(x)$, except at infinity, save that of being summable. The value of $f(x)$ can of course be altered arbitrarily at the points of any set of measure zero, without affecting the values of the integrals (1.1). Two functions which differ only at the points of such a set will be regarded as identical; and when I say that 'the solution is unique' I must be understood to mean that any two solutions are the same for almost all values of x, that is to say except at the points of a set of measure zero.

2. Suppose that $f(x)$ is summable, that the equations (1.1) are satisfied, and that the integral (1.5) is convergent. Then the integral

$$\int_0^\infty |f(x)| \sum \frac{|sx|^n}{2n!}\,dx = \int_0^\infty |f(x)| \cosh \sqrt{|sx|}\,dx,$$

is convergent for $|s| < k$, so that

$$F(s) = \sum_0^\infty \frac{c_n (-s)^n}{2n!} = \int_0^\infty f(x) \sum_0^\infty \frac{(-sx)^n}{2n!}\,dx$$

$$= \int_0^\infty f(x) \cos \sqrt{(sx)}\,dx$$

for $|s| < k$. The integral just written is plainly an analytic function of s, regular for all positive values of s †.

We have therefore

$$F\left(\frac{t^2}{u}\right) = \int_0^\infty f(x) \cos\left\{t\sqrt{\left(\frac{x}{u}\right)}\right\} dx,$$

for all positive values of t and u. Multiply by e^{-t}, and integrate from $t = 0$ to $t = \infty$. Since

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* *Loc. cit.*, p. 387.

† The integral is uniformly convergent throughout any finite domain which lies inside the parabola whose polar and Cartesian equations are

$$r = k^2 \operatorname{cosec}^2 \tfrac{1}{2}\theta, \quad y^2 = 4k^2(x + k^2).$$

On this parabola the imaginary part of $\sqrt{s}$ is equal to k.

$$\int_0^\infty e^{-t}dt \int_0^\infty |f(x)|\,dx$$

is convergent, change of the order of integration is legitimate. We have therefore

$$\int_0^\infty e^{-t} F\left(\frac{t^2}{u}\right) dt = \int_0^\infty f(x)\,dx \int_0^\infty e^{-t} \cos\left\{t\sqrt{\left(\frac{x}{u}\right)}\right\} dt$$

$$= \int_0^\infty \frac{uf(x)}{x+u}\,dx.$$

Thus
$$\chi(u) = \frac{1}{u}\int_0^\infty e^{-t} F\left(\frac{t^2}{u}\right) dt = \int_0^\infty \frac{f(x)}{x+u}\,dx$$

for all positive values of u.

3. The integral last written is plainly an analytic function of u, regular except at points of the negative real axis, which is generally a *coupure*, and the equation

$$\chi(u) = \int_0^\infty \frac{f(x)}{x+u}\,dx \tag{3.1}$$

holds for all such values of u. It was shown by Stieltjes* that, if $f(x)$ is continuous,

$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\{\chi(-\xi - i\epsilon) - \chi(-\xi + i\epsilon)\} = f(\xi), \tag{3.2}$$

when ϵ tends to zero through positive values.

I shall now prove that, *if $f(x)$ is any summable function*, (3.2) *holds for almost all values of* ξ, and in particular for all values of ξ for which

$$X(t) = \int_0^t |f(\xi+\tau) + f(\xi-\tau) - 2f(\xi)|\,d\tau = o(t), \tag{3.3}$$

as in the case for almost all values of ξ.

We have to prove that

$$\int_0^\infty \frac{\epsilon f(x)\,dx}{(x-\xi)^2 + \epsilon^2} \to \pi f(\xi); \tag{3.4}$$

and it is obvious that we may replace the limits of integration by $\xi - \delta$ and $\xi + \delta$, where δ is any positive number, and that we may suppose, without real loss of generality, that $f(\xi) = 0$.

* The idea is due to Hermite. See his memoir 'Sur quelques points de la théorie des fonctions', *Journal für Math.*, vol xci. (1881), pp. 54–78; and his *Cours professé à la Faculté des Sciences*, ed. 4 (1888), pp. 76 *et seq.*, 154 *et seq.*

We have then

$$\int_{\xi-\delta}^{\xi+\delta} \frac{\epsilon f(x)\,dx}{(x-\xi)^2+\epsilon^2} = \int_0^\delta \frac{\epsilon\psi(t)}{t^2+\epsilon^2}\,dt,$$

where

$$\psi(t) = f(\xi+t) + f(\xi-t).$$

Let

$$\Psi(t) = \int_0^t \psi(\tau)\,d\tau,$$

so that

$$|\Psi(t)| \leq X(t) = o(t).$$

Given any positive η, we can choose δ so that

$$X(t) < \eta t$$

for $0 < t \leq \delta$. Then

$$\int_0^\delta \frac{\epsilon\psi(t)}{t^2+\epsilon^2}\,dt = \frac{\epsilon}{\delta^2+\epsilon^2}\Psi(\delta) + \int_0^\delta \frac{2\epsilon t}{(t^2+\epsilon^2)^2}\Psi(t)\,dt.$$

The first integral tends to zero with ϵ, and the second is less than

$$2\eta\int_0^\infty \frac{\epsilon t^2\,dt}{(t^2+\epsilon^2)^2} = 2\eta\int_0^\infty \frac{w^2\,dw}{(w^2+1)^2}.$$

It follows that

$$\int_0^\delta \frac{\epsilon\psi(t)}{t^2+\epsilon^2}\,dt \to 0;$$

and therefore that (3.4) is true.

4. Returning to the analysis of § 2, we see that

$$(4.1)\qquad f(x) = \frac{1}{2\pi i}\lim\{\chi(-x-i\epsilon) - \chi(-x+i\epsilon)\}$$

for almost all values of x, χ being the analytic function defined by the formula

$$(4.2)\qquad \chi(u) = \frac{1}{u}\int_0^\infty e^{-t} F\left(\frac{t^2}{u}\right)dt,$$

and $F(s)$ the function defined by

$$(4.3)\qquad F(s) = \sum_0^\infty \frac{c_n(-s)^n}{2n!}.$$

It follows that $f(x)$ is determined, except at a set of measure zero, by the equations (1.1) and by the condition that the integral (1.5) is convergent. The solution of the problem

* This is the condition from which Lebesgue has drawn so many important consequences.

is therefore, under this condition, and *a fortiori* under the condition (1.3), unique. We have thus proved:

THEOREM A. *There can be at most one solution $f(x)$ of the problème des moments, such that the integral*

$$\int_0^\infty |f(x)|\, e^{k\sqrt{x}}dx$$

is convergent for a positive value of k. This solution of the problem is, if it exists, given by the formulæ (4.1), (4.2), *and* (4.3).

5. Suppose, for example, that c_n is of the form

$$c_n = 2n!\int_0^1 \lambda^n\,\theta(\lambda)d\lambda.$$

Then
$$F(s) = \int_0^1 \frac{\theta(\lambda)}{1+s\lambda}\,d\lambda,$$

and
$$\chi(u) = \int_0^\infty e^{-t}dt\int_0^1 \frac{\theta(\lambda)d\lambda}{t^2\lambda+u} = \int_0^1 \theta(\lambda)\,d\lambda\int_0^\infty \frac{e^{-t}dt}{t^2\lambda+u}$$
$$= \tfrac{1}{2}\int_0^1 \theta(\lambda)\,\frac{d\lambda}{\sqrt{\lambda}}\int_0^\infty \frac{e^{-\sqrt{(\omega/\lambda)}}}{\sqrt{\omega}}\,\frac{d\omega}{\omega+u}.$$

Now
$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_0^\infty \frac{e^{-\sqrt{(\omega/\lambda)}}}{\sqrt{\omega}}\,\frac{2i\epsilon}{\epsilon^2+(\omega-x)^2}\,d\omega = \frac{e^{-\sqrt{(x/\lambda)}}}{\sqrt{x}};$$

and it is easily deduced that

$$\frac{1}{2\pi i}\lim_{\epsilon\to 0}\{\chi(-x-i\epsilon)-\chi(-x+i\epsilon)\} = \frac{1}{2\sqrt{x}}\int_0^1 e^{-\sqrt{(x/\lambda)}}\,\theta(\lambda)\,\frac{d\lambda}{\sqrt{\lambda}}.$$

Thus the solution, if it exists, is

$$f(x) = \frac{1}{2\sqrt{x}}\int_0^1 e^{-\sqrt{(x/\lambda)}}\,\theta(\lambda)\,\frac{d\lambda}{\sqrt{\lambda}};$$

and it is easily verified that this function is in fact a solution. For

$$\int_0^\infty x^n f(x)\,dx = \tfrac{1}{2}\int_0^1 \theta(\lambda)\,\frac{d\lambda}{\sqrt{\lambda}}\int_0^\infty x^{n-\frac{1}{2}}e^{-\sqrt{(x/\lambda)}}\,dx$$
$$= 2n!\int_0^1 \lambda^n\,\theta(\lambda)\,d\lambda = c_n.$$

A similar method may be applied in more general cases, for example when

$$c_n = 2n!\int_0^{a_1}\lambda_1^n\,\theta_1(\lambda_1)\,d\lambda_1\int_0^{a_2}\lambda_2^n\,\theta_2(\lambda_2)\,d\lambda_2\ldots.$$

6. It is sometimes more convenient to use a different form of the solution, valid only under narrower conditions. Let us suppose that there are numbers k and α, where

$$k > 0, \quad \alpha > \tfrac{1}{2},$$

such that the integral

$$\int_0^\infty |f(x)|\, e^{kx^\alpha}\, dx$$

is convergent. Let

$$\mathrm{E}_\alpha(y) = \sum_0^\infty \frac{(-y)^n}{\Gamma(\beta n + 1)},$$

where $\beta = 1/\alpha$. Then it is known* that

$$|\mathrm{E}_\alpha(y)| < K e^{|y|^\alpha},$$

where K is a constant, for all values of y. Hence

$$\int_0^\infty |f(x)| \sum_0^\infty \frac{|sx|^n}{\Gamma(\beta n + 1)}\, dx < K \int_0^\infty |f(x)|\, e^{(|s|x)^\alpha}\, dx,$$

so that the integral on the left-hand side is certainly convergent for sufficiently small values of $|s|$, in fact for $|s|^\alpha < k$. It follows that

$$F(s) = \sum_0^\infty \frac{c_n (-s)^n}{\Gamma(\beta n + 1)} = \int_0^\infty f(x)\, \mathrm{E}_\alpha(sx)\, dx$$

for sufficiently small values of $|s|$. But†

$$|\mathrm{E}_\alpha(y)| < \frac{K}{|y|} \tag{6.1}$$

at all points of a certain region of the type

$$|y| \geq |y_0|, \quad -\delta' \leq am.\, y \leq \delta,$$

which includes the part of the real axis which extends to positive infinity. It follows that the formula

$$F(s) = \int_0^\infty f(x)\, \mathrm{E}_\alpha(sx)\, dx$$

is valid for all positive values of s. Further, we have

$$\chi(u) = \frac{1}{u} \int_0^\infty e^{-t} F\left(\frac{t^\beta}{u}\right) dt = \frac{1}{u} \int_0^\infty f(x)\, dx \int_0^\infty e^{-t}\, \mathrm{E}_\alpha\left(\frac{xt^\beta}{u}\right) dt$$

$$= \int_0^\infty \frac{f(x)}{x+u}\, dx;$$

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* G. Mittag-Leffler, 'Sur la représentation analytique d'une branche uniforme d'une fonction monogène (Cinquième note)', *Acta Mathematica*, vol. xxix. (1905), pp 101–181. See also A. Wiman, 'Ueber den Fundamentalsatz in der Theorie der Funktionen $E_\alpha(x)$', *ibid*, pp. 191–201; and 'Ueber die Nullstellen der Funktionen $E_\alpha(x)$', *ibid.*, pp. 217–234.

In my notation $\mathrm{E}_\alpha(y) = E_{1/\alpha}(-y)$.

† Mittag-Leffler and Wiman, *l.c. supra.*

and $\chi(u)$ is regular for all values of u other than negative values. Repeating the arguments of §§ 3–4, we obtain

THEOREM B. *If there is a solution of the problème des moments such that*

$$\int_0^\infty |f(x)|\, e^{kx^\alpha}\, dx \qquad (k>0,\ \alpha>\tfrac{1}{2})$$

is convergent, it is given, for almost all values of x, by the formulæ

$$f(x)=\frac{1}{2\pi i}\lim_{\epsilon\to 0}\{\chi(-x-i\epsilon)-\chi(-x+i\epsilon)\},$$

$$\chi(u)=\frac{1}{u}\int_0^\infty e^{-t}F\left(\frac{t^\beta}{u}\right)dt,$$

$$F(s)=\sum_0^\infty \frac{c_n(-s)^n}{\Gamma(\beta n+1)},\quad \beta=\frac{1}{\alpha}.$$

It is well to point out explicitly how this method fails when $\alpha<\frac{1}{2}$. In this case the function $\mathrm{E}_\alpha(y)$ is an integral function of real order less than $\frac{1}{2}$, and (to put the matter roughly) becomes exponentially infinite (for appropriate values of $|y|$) when $y\to\infty$ in *any* direction* ; so that there is no such relation as (6.1).

7. It is easy to deduce, from Theorem B, that, if

$$c_n=\Gamma(\beta n+1)\int_0^1 \lambda^n\theta(\lambda)\,d\lambda,$$

where $0<\beta<2$, the solution of the *problème des moments* is given by

$$f(x)=\alpha x^{\alpha-1}\int_0^1 e^{-(x/\lambda)^\alpha}\lambda^{-\alpha}\theta(\lambda)\,d\lambda,$$

where $\alpha=1/\beta$. More generally, we can, by methods similar to those of this section, obtain the solution for

$$c_n=\Gamma(\beta_1 n+\gamma_1)\Gamma(\beta_2 n+\gamma_2)\ldots\int_0^{a_1}\lambda_1^{\,n}\theta_1(\lambda_1)\,d\lambda_1\int_0^{a_2}\lambda_2^{\,n}\theta_2(\lambda_2)\,d\lambda_2\ldots,$$

where β_1, β_2, ... are positive numbers whose sum is less than 2.

A different form of the solution has been given by Le Roy. This I shall discuss in a second note on the subject.

* See A. Wiman, 'Sur une extension d'un théorème de M. Hadamard', *Arkiv för Matematik*, vol. ii. (1905), No. 14, pp. 1–5.

A PROPERTY OF GAUSS-LINES.

By *Prof. E. J. Nanson.*

About ten years ago Mr. W. W. Taylor established results which may be stated as follows:

Given four straight lines, the middle points of the diagonals of the quadrilateral formed by them lie on a line, the Gauss-line of the quadrilateral.

Given five lines, the five Gauss-lines derived from them meet in a point, the Gauss-point of the five lines.

Given six lines, the six Gauss-points derived from them lie on a conic, the Gauss-conic of the six lines.

Given seven lines, the seven Gauss-conics derived from them have in common three points, the Gauss-points of the seven lines.

Given eight lines, the 8×3 Gauss-points derived from them lie on a cubic, the Gauss-cubic of the eight points.

Given nine lines, the nine Gauss-cubics derived from them have in common five points, the Gauss-points of the nine lines.

Given ten lines, the 10×5 Gauss-points derived from them lie on a quartic.

The chain may be continued indefinitely. Thus given $2n$ lines, we have a Gauss $(n-1)$-ic, and given $2n+1$ lines, the $2n+1$ Gauss $(n-1)$-ics derived from them have in common $2n-3$ points, the Gauss-points of the $2n+1$ lines. Also given $2n+2$ lines, the $(2n+2)(2n-3)$ Gauss-points derived from them lie on a Gauss n-ic.

These results were all given by Mr. W. W. Taylor in the *Messenger of Mathematics*, vol. xxxvi., pp. 113-125, the proof being somewhat tedious. Guided by Clifford's proof of his chain I have recently found a simple proof of all these results. But I found that the Gauss $(n-1)$-ics have more than $2n-3$ points in common. They have, in fact, $\frac{1}{2}n(n-1)$ points in common. Thus, given $2n+2$ lines, the $(2n+2) \times \frac{1}{2}n(n-1)$, $= n(n^2-1)$, Gauss-points derived from them lie on a Gauss n-ic.

For simplicity I deal first with the generalised reciprocal properties.

It is well known that the polars of a fixed point, the origin say, with respect to a pencil of conics pass through a fixed point, the conjugate of the origin for the pencil. It is also well known that given five points, the five conjugates of the origin lie on a line, viz. the polar of the origin with respect to the one conic through the five points.

Suppose now that six points are given, and consider the single infinite system of cubics which pass through them and

have a point of inflexion at the origin. It is readily verified, as will be shewn presently, that the harmonic polars of the origin for these cubics touch a conic. But as the cubic may break up into the join of the origin to one of the six points and the conic through the remaining five we see that

Given six points, the polars of the origin with respect to the six conics, each of which passes through five of the six points, touch a conic.

Again, it is easily seen that three cubics of the system pass through a seventh fixed point.

Hence it follows that

Given seven points, the seven conic envelops thereby determined have these tangents in common.

These tangents of course are the harmonic polars of the origin for the three cubics through the seven points.

Suppose now that eight points are given, and consider the singly infinite system of quartics which pass through them and have at the origin a double point with inflexional tangents. It is readily verified that the harmonic polars of the origin for these quartics touch a class-cubic. But as the quartic may break up into the join of the origin to one of the eight points and one of the three cubics which pass through the remaining seven points and have an inflexion at the origin we see that

Given eight points the 8×3 *common tangents derived from them touch a class-cubic.*

Again it is easily seen that six quartics of the system pass through a ninth fixed point.

Hence it follows that

Given nine points the nine cubic envelops thereby determined have six tangents in common.

These tangents of course are the harmonic polars of the origin for the six quartics through the nine points.

This process may clearly be continued indefinitely.

For given $2n$ points, consider the singly infinite system of n-ics which pass through them and have at the origin a multiple point of order $n-2$ with inflexional tangents. The polar $(n-1)$-ic of the origin for such a curve clearly breaks up into the $n-2$ tangents at the origin and a line which may be called the harmonic polar of the multiple inflexional point. It is readily verified that the harmonic polars of this point for the n-ics touch a class $(n-1)$-ic, and it is also readily verified that $\frac{1}{2}n(n-1)$ curves of the system pass through a $(2n+1)^{\text{th}}$ fixed point.

For the equation of any n-ic with a multiple point of order $n-2$ at the origin is of the form

$$A_{n-2} + B_{n-1} + C_n = 0,$$

where A_{n-2}, B_{n-1}, C_n are homogeneous functions of x, y of orders $n-2$, $n-1$, n. Now if the curve pass through $2n$ fixed points we can express the $n+1$ coefficients c_r in C_n and the $n-1$ coefficients a_r in A_{n-2} as linear homogeneous functions of the n coefficients b_r in B_{n-1}. Since the tangents at the origin are inflexional A_{n-2} must be a factor of B_{n-1}. This gives $n-2$ conditions to be satisfied by the $n-1$ ratio of the n coefficients b so that the system of n-ics is singly infinite as already stated. The $n-2$ conditions are

$$\begin{Vmatrix} b_0, & b_1, & b_2, & \ldots, & b_{n-1} \\ a_0, & a_1, & a_2, & \ldots, & 0 \\ 0\ , & a_0, & a_1, & \ldots, & a_{n-2} \end{Vmatrix} = 0 \ldots\ldots\ldots\ldots (1),$$

and if the other factor is $px + qy$, it is clear that $b_0, b_1, \ldots, b_{n-1}$ are bilinear functions of the two sets $a_0, a_1, \ldots, a_{n-2}$; p, q; and that the harmonic polar of the origin is $px + qy + 2 = 0$, this line being, as in the case of the Maclaurin harmonic polar of a point of inflexion on a cubic, the locus of harmonic means of segments of a chord through the multiple inflexion point.

Since the a's are linear functions of the b's it follows that the a's are also bilinear functions of the two sets $a_0, \ldots, a_{n-2}$; p, q. Eliminating the a's from the $n-1$ equations which express this fact, we have at once a non-homogeneous equation of order $n-1$ connecting p and q. It at once follows that the harmonic polar $px + qy + 2 = 0$ of the origin touches a curve of class $n-1$.

Again, if the curve pass through $2n+1$ fixed points we can express the n coefficients b and the $n+1$ coefficients c as linear functions of the $n-1$ coefficients a. Thus the system (1) is* of order $\frac{1}{2}n(n-1)$. Thus through $2n+1$ fixed points $\frac{1}{2}n(n-1)$ curves of the system can be drawn.

Now an n-ic through the $2n$ fixed points having a multiple point of order $n-2$ with inflexional tangents at the origin may break up into the join of the origin to one of the $2n$ fixed points and one of the $(n-1)$-ics, $\frac{1}{2}(n-1)(n-2)$ in number, which pass through the remaining $2n-1$ fixed points, and have a multiple point of order $n-3$ with inflexional tangents at the origin.

* Salmon, *Higher Algebra*, § 271, p. 288.

Hence we see that

Given $2n$ fixed points the harmonic polars of the origin for the $(n-1)$-ics, $\frac{1}{2}n(n-1)(n-2)$ in number, which pass through $2n-1$ of the $2n$ fixed points and have a multiple inflexion of order $n-3$ at the origin touch a curve of class $n-1$.

We also have the theorem that

Given $2n+1$ points the $2n+1$ class $(n-1)$-ics thereby determined have $\frac{1}{2}n(n-1)$ tangents in common.

These tangents are, of course, the harmonic polars of the origin for the $\frac{1}{2}n(n-1)$ n-ics through the $2n+1$ points.

By the successive application of these theorems we see that the chain of results indicated may be continued indefinitely.

By reciprocating these results, and projecting to infinity the line which corresponds to the origin, we obtain the chain of results stated at the commencement of this paper, subject, however, to the modification that the nine Gauss-cubics have, not five, but six common points, and to the modification already stated that the Gauss $(n-1)$-ics have not $2n-3$ but $\frac{1}{2}n(n-1)$ points in common.

As the theorem quoted from Salmon is not proved by him, and is moreover far from easy to prove,* it may be desirable to indicate another way of shewing that $\frac{1}{2}n(n-1)$ of the n-ics pass through $2n+1$ fixed points. This is perhaps the more necessary because the difference between the results of this paper and that of Mr. Taylor depends on the number of these curves.

Since $2n+1$ points are given, the n b's, as we have already seen, are linear functions of the $n-1$ a's. Also, as we have already seen, these are bilinear functions of p, q and of the $n-1$ a's. Equating then the two formulæ for each of the n b's we obtain n equations, linear and homogeneous in the $n-1$ a's, and linear but not homogeneous in p, q. Eliminating then from these n equations the $n-2$ ratios of the $n-1$ a's we get, instead of an $n\times 3$ array as in (1), an $n\times(n-1)$ array, to determine the values of p, q. As the elements of this array are linear in p, q the order of the system for the determination of p, q is $\frac{1}{2}n(n-1)$, as was first shewn by Cayley†. In fact if the order is u_n we readily see that

$$u_n=(n-1)^2-u_{n-1},$$

and as u_2 is obviously 1, this relation gives $u_n=\frac{1}{2}n(n-1)$.

* Cf. *Messenger of Mathematics*, vol. xxxiii. (1903), pp. 33–40.

† *Collected Mathematical Papers*, vol. i., p. 459.

Melbourne,
Nov. 25, 1916.

THE TRIANGLE FORMED BY CIRCULAR ARCS: ITS INSCRIBED, ESCRIBED AND HART CIRCLES.

By *T. C. Lewis, M.A.*

1. Let ABC be any triangle, and $B'C'$ a straight line parallel to BC meeting AB, AC in B', C' respectively, where $AB' = nB'B$. Then a circle can be described to pass through B', C' and to touch the two escribed circles of the triangle opposite the angles B, C; and its radius is given by

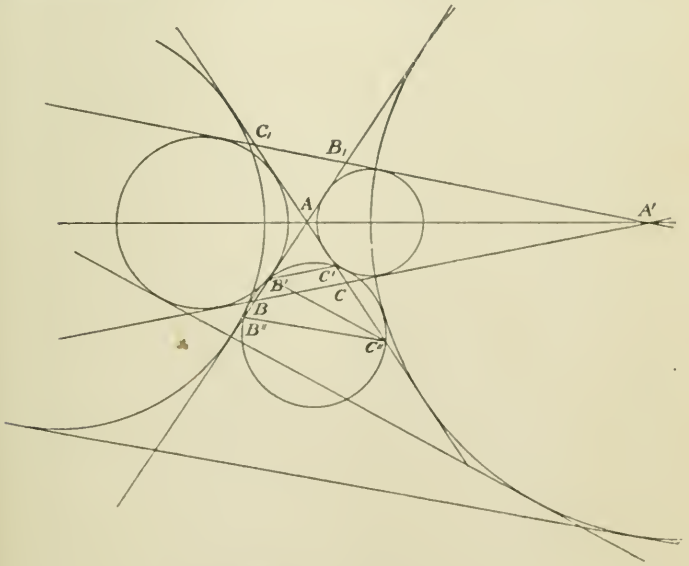
$$4(n+1)\rho = r_1 - r + n^2(r_2 + r_3),$$

$$= \frac{t^2 + n^2}{t} a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $t = \tan \frac{1}{2} A$.

If $n+1$ is negative, the sign of one side of (1) must be changed.

This has already been proved* by the use of tetracyclic co-ordinates. It obviously admits of proof by elementary Trigonometry. This may be effected by the following method.



* *Messenger of Mathematics*, vol. xlvi., pp. 1–23

2. Let O_3, O_2 be the centres of the two escribed circles. Then O_3O_2, BC and the second common external tangent B_1C_1 meet at a point A'. Let A' be the angle between BC and B_1C_1.

If possible let a circle of radius ρ, its centre being at O, be described passing through B', C' and touching the two escribed circles. Let the angle $OB'C'$ be ψ.

Then
$$2\rho\cos\psi=\frac{n}{n+1}a \qquad \text{.....(2),}$$

(but if n and $n+1$ are of different signs, a negative sign must be introduced on one side of this equation).

We have $O_3O=r_3+\rho$, $O_2O=r_2+\rho$.

Also the projection of O_3O on AA' is

$$r_3\sec\tfrac{1}{2}A-\frac{n}{n+1}c\sin\tfrac{1}{2}A+\rho\cos(\psi-\tfrac{1}{2}A'),$$

and on a line perpendicular to AA' it is

$$\frac{n}{n+1}c\cos\tfrac{1}{2}A+\rho\sin(\psi-\tfrac{1}{2}A'),$$

where $A'=C-B$; if C be the greater of the two angles B and C.

Equating the sum of the squares of these to $(r_3+\rho)^2$, we have

$$t^2r_3^2-2r_3\rho-\frac{n}{n+1}r_3\left(2ct-\frac{\cos\tfrac{1}{2}A'}{\cos\tfrac{1}{2}A}a\right)+\frac{n^2}{(n+1)^2}bc\cos A$$
$$+2\rho\sin\psi\left(\frac{\sin\tfrac{1}{2}A'}{\cos\tfrac{1}{2}A}r_3+\frac{n}{n+1}c\sin B\right)=0 \qquad \text{.....(3).}$$

But by projecting BC parallel to, and perpendicular to, AA', it is seen that

$$a\cos\tfrac{1}{2}A'=(b+c)\sin\tfrac{1}{2}A,$$

and
$$a\sin\tfrac{1}{2}A'=(c-b)\cos\tfrac{1}{2}A.$$

Therefore (3) becomes

$$t^2r_3^2-2r_3\rho-\frac{nt}{n+1}(c-b)r_3+\frac{n^2}{(n+1)^2}bc\cos A$$
$$+2\rho\sin\psi\left(\frac{n}{n+1}c\sin B+\frac{c-b}{a}r_3\right)=0.$$

Similarly $O_1O'^2=(r_3+\rho)^2$, and we obtain, writing r_3 for r_2 and interchanging b and c,

$$t^2r_3^2-2r_3\rho+\frac{nt}{n+1}(c-b)\,r_3+\frac{n^2}{(n+1)^2}bc\cos A$$
$$+2\rho\sin\psi\left(\frac{n}{n+1}c\sin B-\frac{c-b}{a}r_3\right)=0.$$

Eliminating $\sin\psi$, we have

$$\left\{t^2r_3^2-2r_3\rho-\frac{nt}{n+1}(c-b)\,r_3+\frac{n^2}{(n+1)^2}bc\cos A\right\}$$
$$\times\left(\frac{n}{n+1}c\sin B-\frac{c-b}{a}r_2\right)$$
$$=\left\{t^2r_2^2-2r_2\rho+\frac{nt}{n+1}(c-b)\,r_2+\frac{n^2}{(n+1)^2}bc\cos A\right\}$$
$$\times\left(\frac{n}{n+1}c\sin B+\frac{c-b}{a}r_3\right).$$

Whence, remembering that

$$t(r_3+r_2)=a,$$

and

$$t(r_3-r_2)=c-b,$$

we obtain

$$4(n+1)t\rho=(t^2+n^2)\,a,$$

and by (2)

$$\cos\psi=\frac{2nt}{n^2+t^2}\quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Hence such a circle can always be found, whatever may be the value of n, *i.e.* whatever the position of $B'C'$; and its radius is determined. Moreover every circle passing through B', C' will touch the escribed circles of some triangle ABC, where BC is parallel to $B'C'$. It will be proved that there are in general two such triangles.

3. A similar proof may be given of the companion proposition, viz. that a circle can be described through $B'C'$, to touch the inscribed circle of the triangle and the escribed circle opposite A, the contacts being of different type; and that the radius ρ is given by

$$4(n+1)\rho=n^2(r_1-r)+r_2+r_3=\frac{n^2t^2+1}{t}a\ldots..(5),$$

It follows that every circle through B', C' will touch the inscribed and the escribed circle opposite A of some triangle

ABC, having the side BC parallel to $B'C'$. It will be found that there are in general two such triangles, the contacts being on the different arcs $B'C'$ of the circle through B', C'.

It may be noted that if (1) and (5) are simultaneously satisfied, $n=1$, and the circle is the nine-point circle. This is Feuerbach's theorem.

4. Consider the equation (1), viz.

$$4(n+1)\rho=\frac{n^2+t^2}{t}a.$$

Here for a given value of ρ there are two circles, determined by the two values of n, such that they touch the two escribed circles indicated. These values, n, n', are connected by the relation

$$nn'+n+n'=t^2,$$

or
$$(n+1)(n'+1)=t^2+1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which is independent of ρ.

We will denote any such circle by the ratio n which determines it; or when convenient we may name any circle by its radius, as (ρ), (r_1), etc.

Now if the circle (n) intersects the sides AB, AC again in $B''C''$, the line joining these two points must be parallel to B_1C_1, and it is seen from the symmetrical position of the circles (n) and (n') on either side of A that

$$AB''=-\frac{n'}{n'+1}AC \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The point A may lie either within or without the circle (n). Suppose it to be outside. The other case may be treated similarly.

There are in addition to the circles (r_2) (r_3) two other circles likewise having AB, AC for internal tangents, and touching the circle (n). They will be the escribed circles of a similar triangle, having the side homologous to AB lying along AC, and *vice versâ*. Let m be the ratio of the homologous sides. Let N, N' be the ratios determining the same circle (n) considered with reference to this latter triangle. Then

$$\left.\begin{aligned} NN'+N+N'=t^2=nn'+n+n' \\ \frac{N}{N+1}m=-\frac{n'}{n'+1}, \text{ and } \frac{N'}{N'+1}m=-\frac{n}{n+1} \end{aligned}\right\} \quad \ldots\ldots\ldots(8)$$

from which m, N and N' may be determined.

There are two solutions, viz.

$$\left.\begin{aligned} & m=-1, \quad N=n', \quad N'=n \\ \text{and} \quad & m=-\frac{nn'}{t^2}, \quad N=-\frac{t^2}{n'}, \quad N'=-\frac{t^2}{n} \end{aligned}\right\} \ldots\ldots(9).$$

The first solution repeats the escribed circles (r_2), (r_3); and the second gives the two other circles, escribed to the triangle similar to ABC. Therefore the sum of the reciprocals of the radii of these four escribed circles is

$$\left(1+\frac{1}{m}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)=\left(1-\frac{nn'+n+n'}{nn'}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)$$

$$=-\left(\frac{1}{n}+\frac{1}{n'}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\ldots\ldots\ldots\ldots(10).$$

5. Consider now equation (5). As before there are two values of n, say n, n', for any value of ρ, and the relation between these roots of the quadratic equation is

$$nn'+n+n'=1/t^2$$

or

$$(n+1)(n'+1)=(t^2+1)/t^2\ldots\ldots\ldots\ldots(11).$$

Let the circle (ρ), which touches (r_2) and (r_3), also touch (mr) and (mr_1), that is the in- and e-scribed circles of a triangle similar to ABC, the homologous sides being taken alternately, as in § 4, and m being their ratio. We have to find m.

Let N, N' be the ratios determining the same circle (ρ), given by equation (1), when considered with regard to this triangle, so that

$$NN'+N+N'=1/t^2.$$

Therefore

$$(NN'+N+N')(nn'+n+n')=1$$

and from the figure

$$\left.\begin{aligned} & \frac{N}{N+1}\,m=-\frac{n'}{n'+1} \\ \text{and} \quad & \frac{N'}{N'+1}\,m=-\frac{n}{n+1} \end{aligned}\right\}\ldots\ldots\ldots(12).$$

There are two solutions, viz.

$$\left.\begin{aligned} & m=n, \quad N=-\frac{n'}{t^2}, \quad N'=\frac{1}{n} \\ \text{and} \quad & m=-n', \quad N=\frac{1}{n'}, \quad N'=-\frac{n}{t^2} \end{aligned}\right\}\ldots\ldots\ldots(13).$$

There are therefore four circles within the angle A, viz. two pairs of circles similar to (r) (r_1), which touch the sides AB, AC and also touch the circle (1). Two lie within and two without that circle. The radii of the two which lie within should be regarded as positive, being the same as for the four escribed circles considered in § 4 and equation (10). The radii of the other two circles, for which there is *one* reversal of contact with AB, AC and the circle (1) relatively to any circle whose radius has been considered positive, must then be regarded as negative.

Thus the sum of the reciprocals of the radii of these four circles is

$$\left(\frac{1}{n}+\frac{1}{n'}\right)\left(\frac{1}{r}-\frac{1}{r_1}\right) \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

Adding this to the sum of the four radii in (10), it follows that the sum of the reciprocals of the radii of the eight circles that touch AB, AC and the circle $B'C'B''$ is zero.

Moreover, if the radii r_2, r_3, mr_2, mr_3, where $m=-\frac{nn'}{t^2}$, as in (9), be multiplied by t^2; and the radii of the four circles within the angle A, viz. mr, mr_1, where m has the two values n, $-n'$, as in (13), be divided by t^2; then the sum for these eight tangent circles

$$= t^2(r_2+r_3)\left(1-\frac{nn'}{t^2}\right)-\frac{1}{t^2}(r_1-r)(n+n')$$

$$= 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

6. We may now invert the figure with regard to any point P, the radius of inversion being k. The triangle $AB'C'$ becomes a triangle formed by three circular arcs; and the eight tangent circles of the first triangle invert into the tangent circles of the second triangle. It may easily be proved that the sum of the reciprocals of the radii of these eight circles is also zero.

For let d be the distance from P of the centre O of any circle of radius r which touches both the straight lines AB, AC, and lies within an external angle at A of the triangle ABC. The reciprocal of the radius of the inverse circle is

$$\frac{d^2-r^2}{k^2 r}=\frac{PA^2}{k^2 r}-\frac{2}{k^2}PA\sec\tfrac{1}{2}A\cos PAO+\frac{t^2}{k^2}r.$$

When this is summed for the four circles concerned the second term vanishes, since it does so for any two circles in vertically opposite angles.

If the circle lies within the angle A of the triangle, the reciprocal of the radius of the inverse circle is

$$\frac{d^2-r^2}{k^2r}=\frac{PA^2}{k^2r}+\frac{2}{k^2}PA\operatorname{cosec}\tfrac{1}{2}A\cos PAO+\frac{1}{k^2t^2}r,$$

where again, on summation, the second term vanishes, two of the radii being positive and the remaining two negative.

And taking the sum for the eight circles, the sum of the first terms vanishes, as also the sum of the third terms as already proved, (15). Therefore in any triangle formed by circular arcs the sum of the reciprocals of the radii of the eight circles which touch all the three circles forming the triangle is zero.

7. We have proved that the radii of the eight in- and e-scribed circles of a triangle formed by two rectilinear sides AB', AC', and by a circular arc $B'C'$ or $C'B'$, are as follows:

escribed opposite B', r_2 and $-\frac{nn'}{t^2}r_2$,

„ C', r_3 and $-\frac{nn'}{t^2}r_3$,

within the angle A, nr and $-n'r_1$,

„ nr_1 and $-n'r$ (to be taken as negative).

We may determine the sides a', b', c' of the rectilinear triangle of which r_2 and $-\frac{nn'}{t^2}r_3$ are the escribed circles, a' being therefore parallel to $B'C''$.

From the figure

$$s'-b'=s-b,\ s'-c'=-\frac{nn'}{t^2}(s-b),$$

and

$$\frac{s'(s'-a')}{s'-c'}=\frac{s'-b'}{t^2}=\frac{s-b}{t^2}.$$

Therefore

$$a'=\frac{n+n'}{t^2}(s-b),\ b'=-\frac{n'(1+n)}{t^2}(s-b),$$

$$c'=\frac{n(1+n')}{t^2}(s-b).$$

Also $$s'=\frac{n}{t^2}(s-b),\ s'-a'=-\frac{n'}{t^2}(s-b),$$

$$s'-b'=s-b,\ s'-c'=-\frac{nn'}{t^2}(s-b),$$

$$\Delta'=-\frac{nn'}{t^3}(s-b)^2.$$

Therefore $$r'=-\frac{n'}{t}(s-b),\ r_1'=\frac{n}{t}(s-b),$$

$$r_2'=-\frac{nn'}{t^3}(s-b),\ r_3'=\frac{1}{t}(s-b).$$

Therefore $$n=\frac{r_1'}{r_3'},\ n'=-\frac{r'}{r_3'},$$

and from (9)

$$N=-\frac{t^2}{n'}=\frac{r_1'}{r_2'},\ N'=-\frac{t^2}{n}=-\frac{r'}{r_2'}.$$

These ratios, n, n', N, N', are four ratios associated with four triangles similar to ABC, any one of which determines the circle $B'C'$.

If n_1, n_1', N_1, N_1' are the like ratios for the other set of triangles with sides a', b', c' or proportional to them, then similarly they are equal respectively to

$$\frac{r_1}{r_3},\ -\frac{r}{r_3},\ \frac{r_1}{r_2},\ -\frac{r}{r_2}.$$

What is most remarkable is that the n-ratios of one set of similar triangles are expressed so simply in terms of the ratios of the radii of the in- and e-scribed circles of the other set.

8. If r_1, r_2, r_3, r be the radii of the escribed and inscribed circles of ABC, then there are in general two circles which touch in like manner the circles of radii r_2, r_3, and also touch in unlike manner two circles within the angle A, touching AB and AC, and having radii mr and mr_1. We have here, in fact, the same equations as in (13), to be used however to determine n, n', &c., in terms of m, instead of the latter in terms of the former.

There are two sets of solutions:

$$n=m,\quad n'=\frac{t^2-m}{m+1},\quad N=-\frac{t^2-m}{(m+1)t^2},\quad N'=\frac{1}{m}$$

and

$$n=\frac{t^2+m}{1-m},\quad n'=-m,\quad N=-\frac{1}{m},\quad N'=-\frac{t^2+m}{(1-m)t^2}\Bigg\}\ldots(16).$$

If one be the circle $B'C'$, the other solution determines another circle touching the four given circles.

9. Such a quartet of in- and e-scribed circles may be taken in eight ways from among the eight circles of that description, whose radii have been determined in §§ 4 and 5.

It is known (Feuerbach's theorem) that there is a circle, viz. the nine-point circle, which touches each of the four in- and e-scribed circles of a rectilinear triangle.

We have now proved that there is a circle, whose radius may be determined, which touches each one of any quartet of such circles, when one of the sides of the triangle is a circular arc, as $B'C'$. Such circle is a Hart circle. And it follows that there are in all eight Hart circles for the complete figure. This property is not altered by inversion, therefore there are also eight Hart circles in the case of a triangle formed by three circular arcs.

To determine the radius ρ' of the Hart circle of any quartet, consisting of the four circles of radius r_2, r_3, mr_1, mr, where $m=n$ as in the first case of (13). The second solution of (10) gives the corresponding Hart circle.

We have

$$4(1+n)\rho=\frac{t^2+n^2}{t}a,$$

and

$$4(1\sim n)\rho'=\frac{t^2+n^2}{t}a.$$

Therefore

$$\rho'=\frac{1+n}{1\sim n}\rho\quad\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

Similarly the seven other Hart radii are found by substituting for n in (17) the values for n', N, N', n_1, n_1', N_1, N_1' already determined in § 7. Now

$$\frac{1+n}{1\sim n}=\frac{r_1'+r_3'}{r_1'-r_3'}=\frac{b'}{a'-c'},\qquad\frac{1+n'}{1\sim n'}=\frac{r_3'-r'}{r_3'+r'}=\frac{c'}{a'+b'},$$

$$\frac{1+N}{1\sim N}=\frac{r_1'+r_2'}{r_1'\sim r_2'}=\frac{c'}{a'\sim b'},\quad\frac{1+N'}{1\sim N'}=\frac{r_2'-r'}{r_2'+r'}=\frac{b'}{a'+c'},$$

and similarly $\frac{1+n_1}{1\sim n_1}=\frac{b}{a\sim c}$, &c., &c.; and the radii of the eight Hart circles are thus determined in a simple form.

10. The method of §§ 1 and 2 leads to the extension of the theorem that the sum of the reciprocals of the radii of the eight tangent circles of three given circles is zero, to include cases in which the three circles do *not* form a triangle, *i.e.* when at least two of them do not intersect, being mutually external.

Two such non-intersecting circles are escribed circles opposite B and C of a triangle ABC. Let the third circle, radius r', have its centre at a distance l from A on a line making an angle ϕ with the bisector of the angle A.

Four of the eight tangent circles will touch the escribed circles opposite B and C in the same way, so that for every one of them the relation between n and ρ is given by (1),

$$4(1+n)t\rho=(t^2+n^2)a.$$

If O be the centre of such a circle, and AO make an angle θ with the bisector of the angle A,

$$OA\cos\theta=\frac{n}{n+1}c\cos\tfrac{1}{2}A+\rho\sin(\psi-\tfrac{1}{2}A').$$

But
$$2\rho\cos\psi=\frac{n}{n+1}a,$$

whence
$$OA\cos\theta=\frac{b+c}{a}\rho\cos(\psi-\tfrac{1}{2}A),$$

and
$$OA\sin\theta=-\frac{n}{n+1}c\sin\tfrac{1}{2}A+\rho\cos(\psi-\tfrac{1}{2}A')$$
$$=\frac{c-b}{a}\rho\sin(\psi-\tfrac{1}{2}A).$$

If this circle also touches the third given circle of radius r',

$$\left\{l\cos\phi-\frac{b+c}{a}\rho\cos(\psi-\tfrac{1}{2}A)\right\}^2$$
$$+\left\{l\sin\phi-\frac{c-b}{a}\rho\sin(\psi-\tfrac{1}{2}A)\right\}^2=(\rho-r')^2\ldots(18),$$

therefore

$$l^2 - r'^2 - 2\frac{l}{a}\rho\{(b+c)\cos\phi\cos(\psi - \tfrac{1}{2}A) + (c-b)\sin\phi\sin(\psi - \tfrac{1}{2}A)\}$$
$$+ \frac{2bc}{a^2}\rho^2\{\cos A + \cos(2\psi - A)\} = -2r'\rho,$$

ρ being positive. This determines two circles. Other two are determined by changing the sign of r', ρ being then reckoned as negative.

Now $$\sin\psi = \frac{n^2 - t^2}{n^2 + t^2}, \quad \cos\psi = \frac{2nt}{n^2 + t^2},$$

therefore

$$\sin(\psi - \tfrac{1}{2}A) = \frac{n^2 - t^2 - 2nt^2}{(n^2 + t^2)\sqrt{(t^2 + 1)}} = \frac{2\rho - at}{2\rho\sqrt{(t^2 + 1)}},$$

therefore

$$2\rho\sin(\psi - \tfrac{1}{2}A) = 2\rho\cos\tfrac{1}{2}A - a\sin\tfrac{1}{2}A,$$

and

$$4\rho^2\cos^2(\psi - \tfrac{1}{2}A) = 4\rho^2\sin^2\tfrac{1}{2}A + 4a\rho\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A - a^2\sin^2\tfrac{1}{2}A,$$

therefore

$$\cos(2\psi - A) = -\cos A + \frac{a}{\rho}\sin A - \frac{a^2}{2\rho^2}\sin^2\tfrac{1}{2}A.$$

Thus

$$\left\{l^2 - r'^2 - \frac{c-b}{a}\,l\sin\phi\,(2\rho\cos\tfrac{1}{2}A - a\sin\tfrac{1}{2}A)\right.$$
$$\left. + \frac{bc}{a}(2\rho\sin A - a\sin^2\tfrac{1}{2}A) + 2r'\rho\right\}^2$$
$$= \frac{(b+c)^2}{a^2}\,l^2\cos^2\phi\,\{4\rho^2\sin^2\tfrac{1}{2}A + 4a\rho\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A - a^2\sin^2\tfrac{1}{2}A\},$$

therefore $$D + (E + Fr')\rho + G\rho^2 = O \quad \ldots\ldots\ldots\ldots(19),$$

where

$$D = \{l^2 - r'^2 + (c-b)\,l\sin\phi\sin\tfrac{1}{2}A - bc\sin^2\tfrac{1}{2}A\}^2$$
$$+ (b+c)^2\,l^2\cos^2\phi\sin^2\tfrac{1}{2}A,$$

and $$F = 4\{l^2 - r'^2 + (c-b)\,l\sin\phi\sin\tfrac{1}{2}A - bc\sin^2\tfrac{1}{2}A\}.$$

Here the sum of the reciprocals of the two values of ρ is

$$-\frac{E + Fr'}{D} \quad \ldots\ldots\ldots\ldots(20).$$

So for the two values of ρ derived from writing $\rho + r'$ instead of $\rho - r'$ in (18), if ρ be taken now as negative, equation (19) becomes

$$D - (E - Fr')\rho + G\rho^2 = O,$$

and the sum of the reciprocals of the roots is

$$\frac{E - Fr'}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$

therefore by addition of (20) and (21), the sum of the reciprocals of the four radii is $-\frac{2Fr'}{D}$.

11. The remaining four tangent circles will touch the escribed circles opposite B and C in different ways. Those escribed circles of the triangle ABC are the inscribed and an escribed circle of the triangle $A'BB_1$. Hence for any of the required tangent circles the relation between n and ρ is given by

$$4(1+n)t'\rho = (n^2t'^2+1)(b+c).$$

Proceeding as before we obtain

$$AO\cos\theta = \frac{a}{b+c}\rho\sin(\psi + \tfrac{1}{2}A'),$$

and

$$AO\sin\theta = d - \frac{a}{c-b}\rho\cos(\psi + \tfrac{1}{2}A'),$$

where

$$d = AA' = \frac{2bc\sin\frac{1}{2}A}{c-b}.$$

Now

$$\cos\psi = \frac{n(b+c)}{2(n+1)\rho} = \frac{2nt'}{n^2t'^2+1},$$

and

$$\sin\psi = \frac{n^2t'^2-1}{n^2t'^2+1},$$

therefore

$$\cos(\psi + \tfrac{1}{2}A') = \frac{2nt' - (n^2t'^2-1)t'}{(n^2t'^2+1)\sqrt{(t'^2+1)}} = \frac{b+c}{2\rho}\cos\tfrac{1}{2}A' - \sin\tfrac{1}{2}A',$$

therefore

$$2\rho\cos(\psi + \tfrac{1}{2}A') = \frac{(b+c)^2}{a}\sin\tfrac{1}{2}A - \frac{2\rho}{a}(c-b)\cos\tfrac{1}{2}A,$$

and

$$4\rho^2 \sin^2(\psi + \tfrac{1}{2}A') = \frac{4\rho^2}{a^2}(b+c)^2 \sin^2 \tfrac{1}{2}A$$

$$+ \frac{4(b+c)^2(c-b)}{a^2}\rho \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A - \frac{(b+c)^4}{a^2}\sin^2 \tfrac{1}{2}A,$$

and

$$4\rho^2\{\cos A' + \cos(2\psi + A')\} = \frac{2(b+c)^4}{a^4}\sin^2 \tfrac{1}{2}A - 4\frac{(b+c)^2(c-b)}{a^4}\rho \sin A.$$

Therefore

$$\left\{l\cos\phi - \frac{a}{b+c}\rho\sin(\psi + \tfrac{1}{2}A')\right\}^2$$

$$+ \left\{l\sin\phi - d + \frac{a}{c-b}\rho\cos(\psi + \tfrac{1}{2}A')\right\}^2 = (\rho + r')^2 \quad \ldots(22)$$

for two of the circles, whose radii are reckoned as positive. For the other two circles the sign of r' is changed, and the radii are to be reckoned as negative. Thus

$$l^2 - r'^2 - 2dl\sin\phi + d^2 + 2(l\sin\phi - d)\frac{a}{c-b}\rho\cos(\psi + \tfrac{1}{2}A')$$

$$- 2l\cos\phi\frac{a}{b+c}\rho\sin(\psi + \tfrac{1}{2}A')$$

$$+ \frac{a^2}{(c^2-b^2)^2}\rho^2\left\{b^2 + c^2 - \frac{(c^2-b^2)^2}{a^2} + 2bc\cos(2\psi + A')\right\} = 2r'\rho \quad \ldots(23).$$

Here the term in ρ^2 is $\dfrac{2a^2bc}{(c^2-b^2)}\rho^2\{\cos A' + \cos(2\psi + A')\}$.

Substituting in (23) the value of $\cos(\psi + \frac{1}{2}A')$, &c., we obtain

$$\left[l^2 - r'^2 - 2dl\sin\phi + d^2\right.$$

$$+ (l\sin\phi - d)\frac{1}{c-b}\{(b+c)^2\sin\tfrac{1}{2}A - 2(c-b)\rho\cos\tfrac{1}{2}A\}$$

$$\left. + \frac{bc}{(c-b)^2}\{(b+c)^2\sin^2\tfrac{1}{2}A - 2(c-b)\rho\cos\tfrac{1}{2}A\} - 2r'\rho\right]^2$$

$$= l^2\cos^2\phi\{4\rho^2\sin^2\tfrac{1}{2}A + 4(c-b)\rho\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A - (b+c)^2\sin^2\tfrac{1}{2}A\},$$

therefore $\qquad D + (E + Fr')\rho + G\rho^2 = O \quad \ldots\ldots\ldots\ldots(24),$

where

$$D=\left\{l^2-r'^2-2dl\sin\phi+d^2+(l\sin\phi-d)\frac{(b+c)^2}{c-b}\sin\tfrac{1}{2}A\right.$$
$$\left.+\frac{(b+c)^2}{(c-b)^2}bc\sin^2\tfrac{1}{2}A\right\}^2+l^2\cos^2\phi\,(b+c)^2\sin^2\tfrac{1}{2}A$$
$$=\{l^2-r'^2+(c-b)\,l\sin\phi\sin\tfrac{1}{2}A-bc\sin^2\tfrac{1}{2}A\}^2$$
$$+(b+c)^2\,l^2\cos^2\phi\sin^2\tfrac{1}{2}A,$$

and $F=-4\{l^2-r'^2+(c-b)\,l\sin\phi\sin\tfrac{1}{2}A-bc\sin^2\tfrac{1}{2}A\}$.

Then the sum of the reciprocals of the roots is $-\dfrac{E+Fr'}{D}$.

So for the two values of ρ derived from writing $\rho-r'$ for $\rho+r'$ in (22), ρ now being taken as negative, equation (24) becomes

$$D-(E-Fr')\rho+G\rho^2=0,$$

and the sum of the reciprocals of the roots is $\dfrac{E-Fr'}{D}$.

Therefore the sum of the four reciprocals is $-\dfrac{2Fr'}{D}$.

But the value of D is the same as in § 10, and the value of F differs only in the sign in the two cases. Therefore the sum of the reciprocals of the eight radii is zero.

12. A further note may be added with reference to the particular case when the three given circles are the escribed circles of the triangle ABC, already considered in the *Messenger of Mathematics*, vol. xlvi., pp. 1 *et seq.*

Three of the eight tangents circles become the sides of the triangle; the nine-point circle makes a fourth; there is also the circle, of radius ρ, which touches the escribed circles internally, *i.e.* so that the centres of circles which touch are on the same side of their common tangent. It has been proved that

$$4r\rho=r^2+s^2.$$

The remaining three circles touch two of the escribed circles externally and the third internally, and their radii ρ_a, ρ_b, ρ_c have been determined by relations of the form $\rho_a=\dfrac{a}{b+c}\rho$.

Now the circle ρ is the inverse of the nine-point circle with respect to the circle which cuts the three escribed circles

orthogonally; the circle ρ_a is the inverse of the side BC; and since the inscribed circle touches the nine-point circle as well as the three sides, its inverse will touch the four circles whose radii are ρ, ρ_a, ρ_b, ρ_c. Let r' be the radius of the inverse of the inscribed circle. Let T be the constant of inversion, viz. the tangent from the radical centre of the three escribed circles to either of them. Then if p_1, p_2, p_3 are the perpendiculars from this radical centre upon the sides of the triangle, the reciprocal of the radius of the inverse of BC is

$$\frac{2p_1}{T^2} = \frac{b+c}{T^2 a} r,$$

therefore

$$\frac{b+c}{a}\frac{1}{\rho} = \frac{b+c}{T^2 a} r,$$

therefore

$$T^2 = r\rho = \tfrac{1}{4}(r^2 + s^2).$$

From this the expression for r', the radius of the circle which touches the circles of radii ρ, ρ_a, ρ_b, ρ_c, may be obtained, viz.

$$\frac{1}{r'} + \frac{1}{r} = \frac{4R}{r\rho}.$$

This circle and the inscribed circle are Hart circles of the system. Similarly the six remaining Hart circles consist of three pairs of inverse circles; three of the radii being $\frac{r_1^2}{r}$, $\frac{r_2^2}{r}$, $\frac{r_3^2}{r}$. Moreover the three circles ρ_a, ρ_b, ρ_c all pass through the radical centre of the escribed circles, and their three remaining points of intersection lie on the circle which is the inverse of the circumcircle of the triangle. If R' be the radius of this circle it may be proved that

$$R' = \frac{2abc\rho}{a^3 + b^3 + c^3 + abc},$$

whence

$$\frac{1}{R} - \frac{1}{R'} - \frac{1}{\rho} = \frac{r}{R\rho}.$$

Similar results follow if the three given circles are the inscribed circle of a triangle and two escribed circles (r_2, r_3), by reversing the sign of the side a, whether it occurs explicitly or in the expression for r, ρ, &c. Thus (r_1) is a Hart circle of the given circle, and of the four circles inscribed and escribed to a triangle each in a Hart circle of the other three.

13. This last result is a particular case of a general proposition. For if any three circles (r_1, r_2, r_3) have a Hart circle (r), the four have a quartet of common tangent circles, and each is therefore a Hart circle of the other three. The four may be called a Hart system.

But any three circles, being mutually external, have eight Hart cirles. Therefore each circle of a Hart system is a choice of one out of eight, *i.e.* there are four groups each containing eight circles, giving thirty-two related circles.

The question arises whether the (r_1) group of eight circles changes, as might be expected, if a different one of the eight Hart circles in the (r) group be chosen to complete the Hart system. If so, we shall be introduced to a much larger family of circles similarly related to the three original circles.

END OF VOL. XLVI.

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